EXPONENTIAL CONVERGENCE FOR THE FREDRIKSON-ANDERSEN ONE SPIN FACILITATED MODEL ON BOUNDED DEGREE GRAPHS

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Abstract. We prove exponential convergence to equilibrium for the Fredrikson-Andersen one spin facilitated model on bounded degree graphs. This was a classical conjecture related to non-attractive spin systems. Our proof rely on coupling techniques based on Harris graphical construction for interacting particle systems.

1. Introduction

intro

Let $G = (V, E)$ be a countable connected graph of bounded degree $\kappa \geq 1$ and let $d : V \times V \to \mathbb{Z}_+$ be the usual graph distance with respect to G. We also denote $x \sim y$, if $x, y \in V$ are nearest neighbor sites ($d(x, y) = 1$). We also require that G contains a copy of $\mathbb{Z}_+$, i.e., there exists $\mathcal{Z} = \{z_i\}_{i\in\mathbb{Z}_+} \subset V$ such that $d(z_i, z_{i+1}) = 1$ for every $i \in \mathbb{Z}_+$. We denote by $\mathcal{G}$ the subgraph $(\mathcal{Z}, \mathcal{E}) \subset G$, where $\mathcal{E}$ is the collection of edges $\{z_i, z_{i+1}\}$, $i \in \mathbb{Z}_+$.

We consider here the Fredrikson-Andersen one spin facilitated model (FA1f) on G which is a continuous time spin system $\eta = (\eta_t)_{t\geq 0}$ with state space $\Omega = \{0, 1\}^V - \{\bar{0}\}$, where $\bar{0}$ is the identically zero configuration, and transition rates $c(\eta, \tilde{\eta})$ equal to zero except for

$$c(\eta, \eta^x) = \begin{cases} \lambda & , \text{ if } \eta(x) = 1 \text{ and } \sum_{y\sim x} \eta(y) > 0, \\ \mu & , \text{ if } \eta(x) = 0 \text{ and } \sum_{y\sim x} \eta(y) > 0, \end{cases}$$

for some $\lambda, \mu > 0$, where η^x is the configuration obtained from η by flipping the spin at site x. We will suppose $\lambda + \mu = 1$, which can be obtained in a standard way by a time rescaling. Then we can fix

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$q = \lambda = 1 - \mu \in (0,1)$ as the unique parameter of the process whose evolution can be informaly described as follows: Each site waits an exponential time of parameter one, independently of any other site, and by this time, if at least one of its neighbors have value one, it takes the value 1 with probability q and the value 0 with probability $1-q$. From now on $q \in (0,1)$ is to be considered fixed.

As usual, regarging configurations in interacting particle system, if $\eta_t(x) = 1$ we will say that site x is occupied by a particle (or simply, that x is an occupied site at time t). Otherwise, we say that site x is empty.

The Bernoulli product measure of parameter q, denoted here by ν_q, is invariant, in fact reversible, for the FA1f process $(\eta_t)_{t\geq 0}$. Other important feature of the FA1f process is that it is not attractive.

In [1] it is discussed the speed of convergence to equilibrium when G is equal to $\mathbb{Z}^d$ with nearest neighbor connections for $d \geq 1$. It is shown (Theorem 2.1 in [1]) that for $q > 1/2$ and initial configurations with sufficiently large and spatially well distributed number of particles, then convergence of the finite dimensional distributions occurs exponentially fast in time with exponent of order $\big(t/\log(t)\big)^{1/d}$. Our aim is to improve their result for q sufficiently close to one by showing an exponential decay to equilibrium with an exponent of order t for a countable connected graph of bounded degree containing a copy of $\mathbb{Z}_+$. So our main result is the following:

theorem:main **Theorem 1.1.** *For q sufficiently close to one, any given site $y \in V$ and every finite dimensional set $\Gamma \subset \Omega$, there exist constants $c > 0$ and $C > 0$ depending on q and Γ such that*

$$\left|\mathrm{P}^{\delta_y}\big(\eta_t \in \Gamma\big) - \nu_q(\Gamma)\right| \leq Ce^{-ct}\,,$$

where δ_y is the configuration with a single particle on site $y \in V$.

Let us start by describing the main steps in the proof of Theorem 1.1 and how they lead to the verification of the statement. Fix $y \in V$ and Γ a finite dimensional subset of Ω, we will also identify it to a subset of $B = B_\Gamma \subset V$ such that Γ only depends on the configuration on sites of B. The main ideia of the proof is to show that we can couple FA1f processes starting at δ_y and ν_q such that, outside an event with probability of order e^{-ct}, the FA1f process starting at δ_y restricted to sites in B has the same configuration at time t as the process starting with distribution ν_q.

The coupling mentioned above is based on the Harris graphical construction of the FA1f process and an associated percolation structure in dual time that allows us to identify activated sites. Let us start by describing the Harris graphical construction: Let $\big(\mathcal{P}_x\big)_{x\in V}$ be a family of rate one Poisson point processes on the half-line $(0,\infty)$ and $(\gamma_{x,n})_{x\in V,\, n\geq 1}$ be family of iid Bernoulli random variables of parameter q which is independent of the Poisson point processes. Then there exists a version of the FA1f process on the same probability space of $\big(\mathcal{P}_x, (\gamma_{x,n})_{n\geq 1}\big)_{x\in V}$ which is defined by

$$\eta_t(x) = \begin{cases} \gamma_{x,n} & , \sum_{y\sim x}\eta_{t-}(y)\geq 1 \text{ and } t\in\big[T_{x,n}, T_{x,n+1}\big)\,,\ n\geq 1\,; \\ \eta_{t-}(x) & , \text{ otherwise}\,. \end{cases}$$

where the $(T_{x,n})_{n\geq 1}$ are the time marks in the Poisson point process $\mathcal{P}_x$, which will also be called *decision times*. For each $x\in V$, we can decompose $\mathcal{P}_x$ in two independent Poisson point processes, one with parameter q associated to points with marks $\gamma_{x,n}=1$, say $\mathcal{P}_x^{'}$, and its complement $\mathcal{P}_x^{''}$. Points in $\mathcal{P}_x^{'}$ will be called *type-1 decision times* and points in $\mathcal{P}_x^{''}$ *type-0 decision times.* We also call

$$\big(\big(\mathcal{P}_x\big)_{x\in V}, (\gamma_{x,n})_{x\in V,\, n\geq 1}\big)$$

the *Harris scheme* for the FA1f model.

Using this definition we obtain a pair of FA1f processes $(\eta_t,\tilde\eta_t)_{t\geq 0}$ starting from any bivariate initial distribution on Ω^2 where both marginals evolve using the same Harris scheme. We are particularly interested in the case where the first marginal starts at δ_y, for some $y\in V$, and the second one starts from the equilibrium measure ν_q. In this case, we represent the probability associated to the process $(\eta_t,\tilde\eta_t)_{t\geq 0}$ by P^{δ_y,ν_q}.

We call a site $x\in V$ an *t-activated* if $\eta_t(x)=\tilde\eta_t(x)$. Our aim is to show that given $x\in V$, then outside an event of exponentially small probability with respect to t, x is t-activated.

Therefore Theorem 1.1 follows from:

prop:main **Proposition 1.2.** *For q sufficiently close to one and every x, $y\in V$, there exist constants $c>0$ and $C>0$ depending on q, x and y such that*

$$\mathrm{P}^{\delta_y,\nu_q}\big(\,x \textit{ is not } t\textit{-activated}\big)\leq Ce^{-ct}\,,$$

for every $t>0$.

To prove Proposition 1.2 we need a proper condition to guarantee that a give site x is t-activated. The main ideia is that x is t-activated if it had the opportunity to choose its spin configuration before time t (at the last possible allowed time) simultaneously for both processes. To use this ideia we need to introduce some definitions and notation. We can define the concept of dual path associated to a given pair

$(x,t) \in V \times [0,+\infty)$ on a given time interval $[0,\tau]$ for some $\tau \in (0,t]$, which we call here a τ-dual path. A τ-dual path of (x,t) is built on a realization of the FA1f process as reversed time piecewise constant rightcontinuous path starting at x such that changes are only possible at decision times. Formally we have $(X(s))_{0<s\le t-\tau}$ that starts at time 0 at position x. It can be constant equal to x or follow the realization of the process backwards in time until a certain decision time $t_1 \in \mathcal{P}_x \cap (\tau,t)$ (if no such point exists then the only possible path is the one that is constant equal to x), then at time $s_1 = t - t_1$ the step function jumps from x to position x_1 chosen among one of its neighbors. By a finite induction procedure, if we have already had k jumps and X is at site x_k at time $s = s_k \le t-\tau$ then, or $X(s) = x_k$ for $s \in (s_k, t-\tau]$, or, if $\mathcal{P}_{x_k} \cap (\tau, t-s_k) \neq \emptyset$, we can take $t_{k+1} \in \mathcal{P}_{x_k} \cap (\tau, t-s_k)$ such that the step function jumps at time $s_{k+1} = t - t_{k+1}$ to a site x_{k+1} chosen among the neighbors of x_k. We denote the random set of all τ-dual paths of (x,t) by $\mathcal{D}(x,t,\tau)$.

For $\tau \in (0,t)$, a path in $X \in \mathcal{D}(x,t,\tau)$ is called an *activated path*, if for some time $s \in (0, t-\tau]$, we have $\eta_{t-s}\big(X(s)\big) = \tilde\eta_{t-s}\big(X(s)\big) = 1$. We denote by $\mathcal{A}(x,t,\tau)$ the random collection of all activated paths in $\mathcal{D}(x,t,\tau)$.

lem:ativo **Lemma 1.3.** *If $\mathcal{D}(x,t,\tau) = \mathcal{A}(x,t,\tau)$ for some $\tau \in (0,t)$, then x is t-activated.*

Proof. The proof follows from a contradiction argument. Suppose that x is not t-activated, we show that there exists a path $X \in \mathcal{D}(x,t,\tau)$ which is not activated. We construct X by a finite number of steps as follows:
Step 1: Since $\eta_t(x) \neq \tilde\eta_t(x)$, then two cases may occur:

(i) $\eta_{t-s}(x) \neq \tilde\eta_{t-s}(x)$ for all $s \in (0, t-\tau)$. In this case the constant path $X \equiv x$ is not t-activated and we stop at step 1.

(ii) $\eta_{t-s}(x) = \tilde\eta_{t-s}(x)$ for some $s \in (0, t-\tau)$. Thus $\mathcal{P}_x \cap (\tau,t) \neq \emptyset$ and we can take $t_1 = \max\big\{r \in \mathcal{P}_x \cap (\tau,t) : \eta_r(x) \neq \tilde\eta_r(x)\big\}$. At time t_1, or $\eta_{t_1}(y) = 0$ for all $y \sim x$ and $\tilde\eta_{t_1}(y) = 1$ for some $y \sim x$, or the same happens exchanging the roles of η and $\tilde\eta$. Thus there exists a neighbor x_1 of x such that $\eta_{t_1}(x) \neq \tilde\eta_{t_1}(x)$. In this case, we consider that X jumps to x_1 at time $s_1 = t - t_1$.

Now by finite induction, we suppose that after Step k, for some $k \ge 1$, we have built our path X up to time $s_k \le t-\tau$ such that $\eta_s\big(X(s)\big) \neq \tilde\eta_s\big(X(s)\big)$ for all $s \in (0, s_k)$. Suppose that $X(s_k) = x_k$ then we perform step $k+1$.

Step k+1: Two cases may occur:

(i) $\eta_{t-s}(x_k) \neq \tilde{\eta}_{t-s}(x_k)$ for all $s \in (s_k, t-\tau)$ and we put $X(s) = x_k$ for $s \in (s_k, t-\tau]$. Then X is not activated and we stop at step $k+1$.

(ii) $\eta_{t-s}(x_k) = \tilde{\eta}_{t-s}(x)$ for some $s \in (s_k, t-\tau)$. Thus $\mathcal{P}_x \cap (\tau, t-s_k) \neq \emptyset$ and we can take $t_{k+1} = \max\big\{r \in \mathcal{P}_{x_k} \cap (\tau, t-s_k) : \eta_r(x) \neq \tilde{\eta}_r(x)\big\}$. At time t_{k+1}, there exists a neighbor x_{k+1} of x such that $\eta_{t_{k+1}}(x) \neq \tilde{\eta}_{t_{k+1}}(x)$. In this case, we consider that X jumps to x_{k+1} at time $s_{k+1} = t - t_{k+1}$.

The number of steps is clearly stochastically dominated by a Poisson distribution of parameter one and then it is finite almost surely. □

From Lemma 1.3, we have that Proposition 1.2 follows from the next result.

prop:main2

Proposition 1.4. *For q sufficiently close to one and every x, $y \in V$, there exist constants $c > 0$ and $C > 0$ depending on q, x and y such that*

$$\mathrm{P}^{\delta_y,\nu_q}\big(\mathcal{D}(x,t,\tau) \neq \mathcal{A}(x,t,\tau)\big) \leq Ce^{-ct},$$

for every $t > 0$.

2. Proof of Proposition 1.4

The proof of Proposition 1.4 is made of three major stages. The first stage is a warming up argument for the process which allows us to guarantee that, outside an event of exponentially small probability, we have an appropriately concentrated and sufficiently large number of occupied sites at time t on $\mathcal{Z}$. The second stage is based on the construction of a percolation struture that will be used in the third stage to show that, also outside an event of exponentially small probability, all dual paths in $\mathcal{D}(x,t,3t/4)$ touchs another path that is capable of transporting ones from time $t/4$. Finally we use the results obtained in the three stages to prove that if the conditions described above for the first and third stages are met then all paths in $\mathcal{D}(x,t,3t/4)$ are t-activated. The idea is to show that all paths in $\mathcal{D}(x,t,3t/4)$ touch some space time point in $V \times (3t/4, t)$ where η and $\tilde{\eta}$ are equal to one and then we need a warming up argument to populate the graph structure for both processes with a sufficiently large number of occupied sites at time $t/4$ (first stage), a suitable percolation structure do define paths that are capable of transporting ones from time $t/4$ to time interval $[3t/4, t]$ (second stage) and a final step to show that we can connect all dual paths in $\mathcal{D}(x,t,3t/4)$ to these tranporting paths. After we have established the three stages described above, we finish the section with the proof of Proposition 1.4.

firststage

2.1. **First Stage.**
We now describe the first stage in the proof of Proposition 1.4. Fix $0 \leq \tau < \tilde{\tau}$, We say that a path $Y : [\tau, \tilde{\tau}] \to \Omega$ is a $(\tau, \tilde{\tau})$-navigated paths, or simply a navigated path, for a FA1f process if

(i) Y is a c.a.d.l.a.g step function;
(ii) $d(Y_s, Y_{s-}) \leq 1$, $s \in [\tau, \tilde{\tau}]$;
(iii) $(\eta_t)_{t\geq 1}$ if $\eta_s\big(Y(s)\big) = 1$ for all $\tau \leq s \leq \tilde{\tau}$.

We are interested in the events $\mathcal{N}((x_0, x_1, ..., x_n), s, t)$ which, for x_0, x_1, ..., $x_n \in V$, $s \geq 0$ and $t > s$, is defined as the event that there exists a $(\tau, \tilde{\tau})$-navigated path for some $s \leq \tau < \tilde{\tau} \leq t$ that starts at site x_0 and visits all sites x_1,...x_n.

So we show first how to construct a navigated path Y from a site $x \in V$, occupied at time τ, to a site $\tilde{x} \in V$. So the process starts at $Y(\tau) = x$. Given Y_t, for some $t > \tau$, the process remains at its current site until the first decision time $t^{'} > t$ in $\mathcal{P}_y$ where y is Y_t or one of its neighbors that are closer to $\tilde{x}$ in graph distance. In the first case, if the spin at site Y_t remains 1 at time $t^{'}$ there is no change of position, otherwise Y jumps to an occupied neighbor that is closest to $\tilde{x}$. In the second case, Y jumps to the neighboring site if it takes value 1. When the process arrives at site $\tilde{x}$ it remains there and do not jump anymore.

To each navigated path Y to a site $\tilde{x}$ starting at site x by time τ we can define the process $S_t = d(Y_t, \tilde{x})$, $t > \tau$, which is a continuous time nearest-neighbor random walk on $\mathbb{Z}_+$ having 0 as an absorbing state that decreases by one at rate greater or equal to q and increases by one at rate smaller or equal to $1 - q$. Thus, by standard large deviations estimates, for $q > 1/2$ the navigated path will quickly arrive at y_0. Indeed there exists a constants c and C depending only on q such that, uniformly over τ,

$$\mathrm{P}\Big(T \geq \frac{d(x, \tilde{x})}{2q - 1} + v \,\Big|\, (\eta_t)_{0\leq t\leq\tau}\Big) \leq Ce^{-ct} \qquad (2.1)$$ fastnavigated

for every $v > 0$, where $T = \inf\{s > 0 : Y_{\tau+s} = \tilde{x}\}$.

lem:navigated

Lemma 2.1. *Let $q > 1/2$ and ν be a initial distribution for the FA1f process satisfying that the distribution of* $\min\{d(x, z_0) : \eta_0(x) = 1\}$ *has exponentially decaying tail. Therefore for every $L > 0$ sufficiently large, there exists $c > 0$, $C > 0$ depending on q, L and ν such that*

$$\mathrm{P}^{\nu}\big(\mathcal{N}((z_0, z_1, ..., z_{Lt}), 0, t)\big) \geq 1 - Ce^{-ct}\,,$$

for every $t > 0$*.*

Remark 2.1. *We can take* $\nu \in \{\nu_q,\ \delta_y,\ y \in V\}$ *is the statement of Lemma 2.1. Clearly* δ_y*, for a fixed* $y \in V$ *satisfies the condition in the statment. For* ν_q*, the random variable* $\min\{d(x, z_0) : \eta_0(x) = 1\}$ *is stochastically dominated by a geometric distribution with parameter* q*, which also implies the condition in the statement.*

Proof. Let $(\eta_t)_{t\geq 0}$ be a FA1f process starting at ν. Take y a random site in V satisfying that $d(y, z_0) = W = \min\{d(x, z_0) : \eta_0(x) = 1\}$. It is clear that

$$\mathrm{P}^{\nu}\big(\mathcal{N}((z_0, z_1, ..., z_{Lt}), 0, t)^c\big)$$

is bounded above by

$$Ce^{-ct} + \sum_{j=0}^{\lfloor \theta t \rfloor} \mathrm{P}^{\nu}\big(\mathcal{N}((z_0, z_1, ..., z_{Lt}), 0, t)^c \big| W = j\big)\mathrm{P}(W = j)\,,$$

for any constant θ such that can be fixed latter. Therefore, we need to show that $\mathrm{P}^{\nu}\big(\mathcal{N}((z_0, z_1, ..., z_{Lt}), 0, t)^c \big| W = j\big)$ decays exponentially fast as t goes to infinity uniformly for $j \in \{0, 1, 2, ..., \lfloor \theta t \rfloor\}$.

Now fix j as above and an occupied site at time 0, $y \in V$, such that $d(y, z_0) = j$. We have that $\mathcal{N}((z_0, z_1, ..., z_{Lt}), 0, t)$ happens if we build a concatenation of $Lt + 1$ navigation paths between the pairs (y, z_0), (Z_0, z_1), ... , (z_{Lt-1}, z_{Lt}) such that the time length of the concatenated path is smaller than t. so we build these paths using the construction described above and denote their time length by T_1, ... , T_{Lt+1}. By the Strong Markov property, these are independent random variables whose distribution is stochasticaly dominated by the absorbing time of a homogeneous positive recurrent nearest neighbor continuous time random walk on $\mathbb{N}$ (see above). Moreover the large deviations estimates 2.1 implies that these times have some exponential moments and by Crámer Theorem for L sufficiently large

$$\mathrm{P}\Big(\sum_{j=1}^{Lt+1} T_j \geq t\Big) \leq Ce^{-ct}\,.$$

□

We finish the first stage by using Lemma 2.1 and a comparison with a discrete time contact process to control the number of occupied sites among $\{z_0, z_1, , ..., z_{Lt}\}$ at time t.

We will use the Harris scheme to couple $(\eta_t)_{t\geq 0}$ to a discrete time contact process $(\xi_n)_{n\geq 0}$ which is a discrete Markov Process with state space $\{0,1\}_+^{\mathbb{Z}}$ such that given ξ_n we have that $(\xi_{n+1}(j))_{j\in\mathbb{Z}_+}$ are condicionally independent and, for some $p,\hat{p}\in(0,1)$,

$$\mathrm{P}\big(\xi_{n+1}(j)=1\big|\xi_n\big)=\left\{\begin{array}{cl} p & \text{, if } \xi_n(j)=1\,, \\ 1-(1-\hat{p})^{\eta_n(1)} & \text{, if } j=0,\ \xi_n(j)=0\,, \\ 1-(1-\hat{p})^{\eta_n(j+1)+\eta_n(j-1)} & \text{, if } j>0,\ \xi_n(j)=0\,. \end{array}\right. \quad (2.2)$$

lem:coupling

Lemma 2.2. *For q sufficiently close to one and $\theta>0$ sufficiently large, there exists $p=p(q,\theta)$ and $\hat{p}=\hat{p}(q,\theta)$ in $(0,1)$ and a coupling between the the FA1f process, $(\eta_t)_{t\ge0}$, and a discrete contact process of parameters p and $\hat{p}$, $(\xi_n)_{n\ge0}$, such that if $\eta_0(z_m)=1$ and ξ_0 is the configuration on $\{0,1\}_+^{\mathbb{Z}}$ with a single particle at m then almost surely for all $j\in\mathbb{Z}_+$, $\xi_n(j)=1$ implies that $\eta_{\theta n}(z_j)=1$. Futhermore, $p(q,\theta)\to1$ and $\hat{p}(q,\theta)\to1$ as $q\to1$ and $\theta\to\infty$.*

Remark 2.2. *Although we lose information when we replace the FA1f process by the discrete contact process, which should be clear by the proof of Lemma 2.2, we need it due the lack of attractivity of the FA1f and the need to have some proper estimates on the density of ones by time t. Moreover, we can rely on the fact that the discrete contact process is well known, see from instance Durrett* [2, 3]*. On Section 2.2 we discuss another discrete (but dual) time contact process and we recall some properties of such processes.*

Proof. We will consider a version of $(\xi_n)_{n\ge0}$ built using the Harris scheme for the FA1f process. We consider a partition of the time interval into disjoint consecutive intervals of length θ. So considering the values of $\eta_{\theta n}$ on $\mathcal{Z}$ and ξ_n and supposing that $\eta_{\theta n}(z_j)\ge\xi_n(j)$ for every $j\ge1$, we want to use the restriction of the Harris scheme to the time interval $(\theta n,\theta(n+1)]$ to specify ξ_{n+1} such that we still have $\eta_{\theta(n+1)}(z_j)\ge\xi_{n+1}(j)$ for every $j\ge1$. Once this specification is done the proof follows from induction.

We have to obtain the parameters p and $\hat{p}$ in the definition of the transition probabilities in (2.2). Put $\xi_0=\eta_0$ and fix $j\ge1$. To obtain $\xi_{n+1}(j)$ from ξ_n using the Harris scheme define $W_k^{'}$ as the waiting time from θn to the first occurence of a time in $\mathcal{P}_{z_k}^{'}$, i.e.

$$W_k^{'}=\min\{\mathcal{P}_{z_k}^{'}\cap[\theta n,\infty)\}-\theta n\,,$$

and $W_k^{''}$ is defined analogously using $\mathcal{P}_{z_k}^{''}$.

we only need to consider the three complementary cases below:

Case $\eta_{\theta n}(z_j)=\xi_n(j)=1$:

Here if $\mathcal{P}''_{z_j} \cap [\theta n, \theta(n+1)] \neq \emptyset$ then $\eta_{\theta(n+1)}(z_j) = 1$. This happens with probability

$$p' = \mathrm{P}\big(W''_j > \theta\big) = e^{-\theta(1-q)}\,.$$

Thus we simply fix $p = p'$.

Case $\xi_n(j) = 0$ with $\xi_n(j \mp 1) = 0$ and $\eta_{\theta n}(z_{j\pm1}) = \xi_n(j \pm 1) = 1$:

Suppose $\xi_n(j-1) = 0$ and $\eta_{\theta n}(z_{j+1}) = \xi_n(j+1) = 1$, the other case is analogous. If $\xi_{n+1}(j) = 1$ we should have $\eta_{\theta(n+1)}(z_j) = 1$ which happens in the event

$$\{W''_j > \theta\} \cap \big\{W'_j < \big(\theta \wedge W''_{j+1}\big)\big\}\,.$$

By a standard computation, the probability of this previous event is equal to

$$p'' = qe^{-\theta(1-q)}(1 - e^{-\theta})\,.$$

Then we should have $\hat{p} \geq p''$.

Case $\xi_n(j) = 0$ with $\eta_{\theta n}(z_{j-1}) = \xi_n(j-1) = \eta_{\theta n}(z_{j+1}) = \xi_n(j+1) = 1$: In this case, to guarantee that $\xi_{n+1}(j) = 1$ implies $\eta_{\theta(n+1)}(z_j) = 1$ we use the event

$$\{W''_j > \theta\} \cap \big\{W'_j < \big(\theta \wedge (W''_{j-1} \vee W''_{j+1})\big)\big\}\,.$$

Its probability can be computed explicitly as

$$p''' = qe^{-2\theta(1-q)}\left[2(1 - e^{-\theta}) - \frac{1 - e^{-\theta(2-q)}}{(2-q)}\right]\,.$$

We also should have $\hat{p} \geq 2p''' - (p''')^2$.

From the second and third cases above, it is enough to take $\hat{p} = \max\{p'', 2p''' - (p''')^2\}$. Finally it is clear from the definitions that $p(q,\theta) \to 1$ and $\hat{p}(q,\theta) \to 1$ as $q \to 1$ and $\theta \to \infty$. □

lemma:densidade

Proposition 2.3. *For each $\rho \in (0,1)$ and L sufficiently large, if q is sufficiently close to one, there exists $c > 0$ and $C > 0$ depending on q, ρ and L such that*

$$\mathrm{P}\Big(\frac{\#\big\{j \in \{0,1,...,Lt\} \;:\; \eta_t(z_j) = 1\big\}}{Lt} \leq \rho\Big) \leq Ce^{-ct}\,,$$

for every $t > 0$.

Proof. Apply Lemma 2.1 considering navigated paths on time interval $[0, t/2]$ and we have that there exists $L > 0$ such that

$$\mathrm{P}^{\nu}\big(\mathcal{N}\big((z_0, z_1, ..., z_{Lt}), 0, t/2\big)^c\big) \leq Ce^{-ct}\,.$$

So we only need to show that given $\mathcal{N}\big((z_0,z_1,...,z_{Lt}),0,t/2\big)$, the probability of

$$\Big\{\frac{\#\big\{j\in\{0,1,...,Lt\}\;:\;\eta_t(z_j)=1\big\}}{Lt}\le\rho\Big\}$$

decays exponentially fast if q is sufficiently large.

Now we are going to use the coupling with the discrete time contact processes and a small renormalization argument. Let us fix $R>0$ that should be taken large. We make a parttition of $\{0,1,...,Lt\}$ into the sets $\Gamma_l=\{(l-1)R,...,lR-1\}$, $1\le l\le\lceil(Lt+1)/R\rceil$. For $\alpha\in(0,1)$ let W_l^α be Bernoulli random variables defined as follows: $W_l^\alpha=1$ if the number of occupied sites in Γ_l by time t is bounded below by αR, otherwise $W_l^\alpha=0$.

Recall that we are conditioning on $\mathcal{N}\big((z_0,z_1,...,z_{Lt}),0,t/2\big)$ and each set Γ_l has an occupied site during some time in the interval $[0,t/2]$. Put $p_\alpha=P\big(W_l^\alpha=1\big)$. From section 8 and 14 in [3], it follows that p_α can be as close to one as necessary by taking R sufficiently large, as far as p and $\hat{p}$ are both greater then the critical probability for the dicrete time contact process and α is smaller than $\mathrm{P}(0\in\xi_\infty^{\mathbb{Z}_+})$. Note that $\lim_{p,\hat{p}\to1}\mathrm{P}(0\in\xi_\infty^{\mathbb{Z}_+})=1$, see section 14 in [3] (We remark that the percolation structure from [3] is not exactly the one associated to the discrete contact processes, but the results remain valid with some straightforward adaptation of the arguments there, see also [3, 4]). Morever, from the FKG inequality, we have that $P(W_l^\alpha=1|W_k^\alpha=1)\ge P(W_l^\alpha=1)=p_\alpha$, for every $1\le l,k\le\lceil(Lt+1)/R\rceil$. Therefore from Theorem 0.0 in [5], if $p_\alpha>3/4$ then we have that the $W_l's$ are stochastically dominated from below by iid Bernoulli random variables of parameter $\tilde{p}_\alpha=\tilde{p}_\alpha(q,\theta,R)$ such that $\lim\tilde{p}_\alpha=1$ as $q\to1$, $\theta\to\infty$ and $R\to\infty$.

Now from the Large deviations for iid Bernoulli random variables, for each $\epsilon>0$ fixed, outside an event of exponentially small probability

$$\sum_{l=1}^{\lceil(Lt+1)/R\rceil}W_l^\alpha\ge(\tilde{p}_\alpha-\epsilon)\Big\lceil\frac{Lt+1}{R}\Big\rceil,$$

ou ainda

$$\frac{\#\big\{j\in\{0,1,...,Lt\}\;:\;\eta_t(z_j)=1\big\}}{Lt}\ge\alpha(\tilde{p}_\alpha-\epsilon)\,.$$

Now, simply choose α, R and ϵ such that $\alpha(\tilde{p}_\alpha-\epsilon)>\rho$ to finish the proof. □

secondstage

2.2. **Second Stage.**

In this section we define a percolation structure based on the Harris graphical construction of the FA1f and on discrete time contact processes, similar to and related to processes considered in Section 2.1. We will be motivated by trying to understand the dual process of our FA1f model $(\eta_t)_{t\geq 0}$.

From this time $t > 0$ will be considered fixed. Given $p_0 \in (0,1)$ (and for our purposes close to 1) let $K = -2log((1-p_0)/2)$, so the probability of having at least a decision point in an interval of length $K/2$ is equal to $(1+p_0)/2$. Now we renormalize time and discretize space time via (dual) intervals

$$I(y,i) = \{y\} \times [iK/2, (i+1)K/2] \subset V \times \mathbb{Z}_+ .$$

We say that (y,i) (or equivalently $I(y,i)$) is *good* if the following two conditions hold:

(i) In the Harris scheme the interval $\{y\} \times [t-(i+1)K/2, t-iK/2]$ contains no type-0 decision point.

(ii) In the Harris scheme the interval $\{y\} \times [t-(i+2)K/2, t-(i+1)K/2]$ contains at least one type-1 decision point.

The importance being that if we are given sites $y = y_0,\, y_1,\, \cdots y_m$ in V with $y_i \sim y_{i-1}$, for every $i = 1, ..., m$, then if (y_i, i) is good for each i and $\eta_{t-(mK)}(y_m) = 1$, it follows that $\eta_t(y) = 1$.

We now define the one dimensional contact process mentioned above. Recall the definition of $\mathcal{Z} = \{z_0, z_1, z_2, \dots\}$ from section 1 and fix $y_0 \in V$. Let $y_0, y_1, \dots, y_r = z_j$ be the shortest path from y_0 to $\mathcal{Z}$. We let $\mathcal{Z}^{y_0}$ be the copy of $\mathbb{Z}_+$

$$y_0, y_1, \dots, y_r, z_{j+1}, z_{j+2}, \dots$$

Let us suppose for the moment that y_0 and $k \geq 0$ are fixed. For w in $\mathcal{Z}^{y_0}$ and $l \geq 0$ consider Bernoulli random variables $J_k(w,l)$ which are equal to one if and only if $I(w, k+l)$ is good. Then the random variables $J_k(w,l)$ are independent of all other $J_k(u,l')$ random variables except $u = w$ and $|l-l'| = 1$. We derive our contact process" $\xi^{y_0,k}$ by

$$\xi_0^{y_0,k}(x) = \delta_{y_0}(x) := \begin{cases} 1 & , \ x = y_0 \\ 0 & , \ \text{otherwise,} \end{cases}$$

and

$$\xi_n^{y_0,k}(x) = 1 \text{ if and only if } J_k(x,n) = 1 \text{ and } \xi_{n-1}^{y_0,k}(w) = 1$$

for w equal to x or neighbouring it.

To motivate this process note that if for some $n \geq 1$ and $w \in \mathcal{Z}^{y_0}$ we have that $\eta_{t-(i+n+2)K/2}(w) = 1$ and $\xi_n^{y_0,k}(w) = 1$, then $\eta_{t-iK/2}(y_0) = 1$.

Our process is very similar to previously discussed contact processes. An annoying difference is that the Bernoulli random variables are not independent. However we will be primarily concerned with increasing events and by applying [5] (or using simple contour arguments) we can (by lowering infection parameter p) assume that $\left(\xi_n^{y_0,k}\right)_{n\geq 0}$ is the fully independent model.

Our overall aim is to show that if the contact process dies out, the die out time has exponential moments for small parameter and that if the process survives it must give many occupied sites.

Here we simply record some simple properties for discrete time "contact processes" having infection parameter p sufficiently close to 1. The contact processes will be on half lines rather than on $\mathbb{Z}$ as in our graph, since we are guaranteed half lines but not necessarily copies of $\mathbb{Z}$.

We now list some elementary but useful properties of $\xi^{y_0,k}$. In fact we will drop y_0, k from the notation $y_0 k$ and consider $(\xi_n)_{n\geq 0}$ discrete time contact processes on $\mathbb{Z}_+$ with a variety of (non zero) initial conditions. This is the same notation used in Section 2.1, although the contact processes are not the same. There is no prejudice since the contact process of Section 2.1 is not used outside that section, moreover the results we state below hold in both cases. For the proofs and more on contact processes we suggest [3] and [4].

We need some notation before continuing. Put $\nu = \inf\{n \geq 0 : \xi_n \equiv 0\}$ and $r_n = \sup\{x \in \mathbb{Z}_+ : \xi_n(x) = 1\}$.

death **Proposition 2.4.** *There exists $c > 0$ so that uniformly over ξ_0*

$$E(e^{c\nu}, \nu < \infty) \leq 2.$$

Furthermore as the infection probability p tends to one, c can be allowed to become arbitrarily large.

death2 **Proposition 2.5.** *For each $\beta < 1$ there exists $p_\beta < 1$ so that for all $p \in [p_\beta, 1]$ and $y_0 \in \mathbb{Z}_+$ if $\xi_0 = \delta_{y_0}$ then for every $n > 0$*

$$P\big(r_n < \beta n + y, \nu > n\big) \leq (p_\beta)^n .$$

The latter proposition can be pushed to the following result.

Proposition 2.6. *For every $0 < R < 1$, there exists $\tilde{p} < 1$ so that, for every $|y_0| \leq Rt$ and $p \in [\tilde{p}, 1]$, if $\xi_0 = \delta_{y_0}$ then*

$$P\Big(\{\nu > 2Rt\} \cap \big\{\exists\, m \geq Rt: \; r_m < \frac{Rt}{2} \text{ or } \xi_s(0) = 0 \;\forall\, s \in [Rt,\, 2Rt]\big\}\Big)$$

is bounded above by $\tilde{p}^{2Rt}$.

Corollary 2.7. *For every $0 < R < 1$, there exists $\tilde{p} < 1$ so that, for every $|y_0| \leq Rt/2$ and $p \in [\tilde{p}, 1]$, if $\xi_0 = \delta_{y_0}$*

$$P\Big(\exists\, n \in (2Rt, t) \text{ with } \sum_{0 \leq x \leq \frac{Rt}{2}} \xi_n(x) < \frac{9Rt}{20}, \nu \geq 2Rt\Big)$$

is bounded above by

$$t\, P\Big(\sum_{0 \leq x \leq \frac{Rt}{2}} \hat{\xi}(x) < \frac{9Rt}{20}\Big) + \tilde{p}^{2Rt}$$

where $\hat{\xi}$ is a configuration in non trivial equilibrium.

We now relate these results to our discrete time process $\left(\xi_n^{y_0,k}\right)_{n \geq 0}$. We will be interested in two contact processes. The original process on $\mathcal{Z}^{y_0}$ and a related "subordinate" process on $\mathcal{Z}$ itself.
Recall that $y_0 \in V$ and $k \geq 1$ are fixed. We first note that if ν, the death time for $\xi_n^{y_0,k}$, is greater than Rt then outside of probability e^{-ct} we have (for $|y_0| < \frac{Rt}{2}$) that $\xi_n^{y_0,k}$ is not empty on $\mathcal{Z} \cap \mathcal{Z}^{y_0}$ $\forall n \geq Rt$. We now consider (following [2]) the stopping times $\nu_0, \nu_1, \ldots$ defined as follows $\nu_0 = Rt$ at this time pick a site $y = y_1$ in $\mathcal{Z}$ for which $\xi_{\nu_0}(y) = 1$. Let ν_1 be the (possibly infinite) time when the contact process beginning at ν_0 with only y occupied on $\mathcal{Z}$ expires. Given ν_{i-1} let y be replaced by a new site y_i in $\mathcal{Z}$ so that $\xi_{\nu_i-}(y_i) = 1$ and let ν_i be the (possibly infinite) time that the discrete time contact process in $\mathcal{Z}$ starting at ν_{i-1} dies. The following is a simple consequence of Propositions 2.4 and 2.5.

Lemma 2.8. *For every $0 < R < 1$, there exists $c > 0$ and $\tilde{p} \in (0, 1)$ so that for $|y_0| \leq \frac{Rt}{2}$ and $p > \tilde{p}$*

$$P\big(\{\nu \geq Rt\} \cap E\big) \leq e^{-ct},$$

where

$$E = \{\text{For some choice of } y_1, y_2, \ldots \text{ there is no } i < 2Rt \text{ with } \nu_i = \infty\}.$$

From this we immediately obtain

prop:percolation

Proposition 2.9. *For every $0 < R < 1$, there exists $\tilde{p} < 1$ and $c > 0$ so that, for every $|y_0| \leq Rt/2$, $|k| \leq Rt/2$ and $p \in [\tilde{p}, 1]$*

$$P\Big(\sum_{j=0}^{\frac{Rt}{2}} \xi^{y,k}_{Rt-k}(j) < \frac{9}{20}Rt, \nu \geq Rt\Big) \leq e^{-ct}.$$

thirdstage

2.3. Third Stage.

Recall the definition of dual paths and $\mathcal{D}(x,t,\tau)$ from Section 1. Here $x \in V$ is a fixed site which is at (graphical) distance Rt from our "origin" z_0. We are interested in paths in $\mathcal{D}(x,t,3t/4)$. We say a dual path $X \in \mathcal{D}(x,t,3t/4)$ *encounters a good percolating interval* $I(y,i)$ if for some $s \in [t - iK/2, t-(i+1)K/2]$, $X(t-s) = y$.

The objective of this section is to show the following result:

hitperc

Proposition 2.10. *Let $0 < R < 1$ and $K > 0$ be fixed as in the previous section. There exists $q_0 < 1$ so that for $q > q_0$ and all t large if $|y| \leq Rt$ fixed, the probability that there exists a dual path starting at (y,t) which does not encounter a K normalized "dual" contact process that survives until time $t/4$ is less than e^{-ct} for some $c > 0$.*

In analyzing dual paths we will use various codings (or discrete representations for these objects. We begin with a basic coding. A dual path can be coded (in 1-1 fashion) by a sequence $y = y_0, y_1 \cdots y_m$ where $\forall i$, y_i and y_{i-1} are either equal or nearest neighbours and so that if we define times t_i recursively by $t_0 = 0$ and for $i > 0$,

$$t_i \;=\; \inf\{s > t_{i-1} : (y_{i-1}, t-s) \text{ is a decision point }\},$$

then $X(s) \;=\; y_i$ on $[t_i, t_{i+1})$ and $t_{m+1} > t$. The "value" of X, m, is denoted by $|X|$.

lemcount1

Lemma 2.11. *For every sufficiently large N, we have that*

$$P(\exists\ X \in \mathcal{D}(y_0,t,3t/4) \textit{ with } |X| > Nt) \leq e^{-t}.$$

Proof. The statement of the lemma is that we cannot find $y_0, y_1 \cdots y_{Nt}$ such that for all i, y_i and y_{i-1} are either equal or nearest neighbours and (with the above definition) $\sum_{i=1}^{Nt}(t_i - t_{i-1}) \;\leq\; t/4$. Now there are (at most) $(\kappa+1)^{Nt}$ (recall that κ is the degree of the graph) such sequences and the probability that for any such fixed sequence has $\sum_{i=1}^{Nt}(t_i - t_{i-1}) \;\leq\; t/4$ is equal to the probability that $\sum_{i=1}^{Nt} e_i \;\leq\; t/4$ for i.i.d. standard exponential random variables e_i. So by standard

Tchebychev bounds the probability in the statement is bounded above by

$$\frac{\big((\kappa+1)E(e^{-(\kappa+1)\,e_1})\big)^{Nt}}{e^{-(\kappa+1)\,t/4}} \;=\; \Big(\big(\frac{k+1}{k+2}\big)^N e^{(\kappa+1)/4}\Big)^t \;\leq\; e^{-t}$$

for N large and all t positive. □

We now consider a coding of a dual path X which is "compatible" with the discretization imposed by the renormalization procedure of Section 2.2. Given the coding $y = y_0, y_1, \cdots y_m$ (given Lemma 2.11 we may and shall assume that $m < Nt$), we define a skeleton of it $(v_1, v_2 \cdots v_{t/(2K)})$ to be such that for all i in time interval $[(i-1)K/2, iK/2]$, the path X begins at a site z_a^i and ends at site z_b^i which are linked by a path of v_i sites each visited by X in this interval. An interval may correspond to several skeletons. We denote by $\{y_i\}\{v_j\}$ a pair where $\{y_i\}$ is a code and $\{v_j\}$ is its associated skeleton.

lemcount2 **Lemma 2.12.** *(i) There are at most* $\sum_{L=0}^{N\,t} \binom{L+t/(2K)}{t/(2K)} \leq C_1^t$ *choices of skeleton corresponding to dual paths of size less than* Nt *for* $C_1 > 0$ *not depending on* t.
(ii) Given $\tilde v = (v_1, v_2, \cdots, v_{\frac{t}{4K}})$ *there are at most* C_2^t *choices of corresponding codes for some* $C_2 > 0$ *not depending on* t.

Proof. We note first that $\sum_i v_i \;\leq\; Nt$ and if L is the sum, the number of skeletons is exactly $\binom{L+t/(2K)}{t/(2K)}$. By summing over L we can get an upper bound of $2^{Nt+1+t/2K}$ and inequality (i) follows. Part (ii) follows from the standard path counting which gives at most $(\kappa+1)^{\frac{t}{2K}} = C_2^t$ corresponding codes.

□

Thus every dual path X we have a renormalized coding $\{y_i\}\{v_j\}$ given by

$$(y_0 \cdots y_{v_1})\,,\ (y_{v_1} \cdots y_{v_1+v_2})\,,\ \ldots\,,\ (y_{v_1+\ldots+v_{(t/2K)-1}} \cdots y_{v_1+\ldots+v_{t/2K}})\,.$$

For instance e.g. $y_{v_1+v_2} \cdots y_{v_1+v_2+v_3}$ represents a v_3 path of visited sites from the first visited site to the last on the third time interval. Unlike the initial coding there are multiple renormalized codings for a given X but by Lemma 2.12 in any case there are at most $(C_1C_2)^t$ such codings.

lembound1 **Lemma 2.13.** *For* $0 < \epsilon < 1$ *and fixed renormalized coding, the probability that more than* ϵt *of the intervals visited are bad is less than* $e^{-\theta t}$ *uniformly over* y_0 *chosen, where* $\theta = \theta(q) \uparrow \infty$ *as* $q \uparrow 1$ *and* $K \uparrow +\infty$.

In particular we can take q sufficiently large that $C_1^t C_2^t e^{-\theta(q)t}$ is exponentially small.

Proof. Let us simply remark that the intervals at a fixed time level are independent, while given the information on thestatus's up to (dual) time $(i+1)K/2$, the status of $I(y_j,i)$ are conditionally independent for $y_j \in y_{v_1+v_2\cdots v_i}\cdots y_{v_1+v_2+\cdots y_{v_{i+1}}}$ with $P(I(y_j,i)$ is good $|\mathcal{F}_i) \geq e^{(1-q)K/2}(1-e^{-qK/2})$ if either $I(y_j,i-1)$ is not identified or is good. Thus we easily see our probability is bounded by the probability that a binomial with parameters Nt and $1-\ e^{(1-q)K/2}(1-e^{-qK/2})$ has value greater than $\epsilon/2$. The desired bound now follows from elementary binomial tail bounds. □

Since we are interested in the event that some dual path never encounters a good interval which percolates for time $t/4$. Were this to happen then some renormalized code would never encounter a good interval which percolates. Then every interval encountered would either be bad (which by Lemma 2.13 for large enough q would only be a small proportion) or must have a finite percolation lifetime. Thus (unless the bound of Lemma 2.13 is violated) we must be able to find a collection of levels $i_1, i_2, \cdots i_f$ and associated to each level i_j a $w_j \in y_{v_1+v_2\cdots v_{i_j}}\cdots y_{v_1+v_2+\cdots y_{v_{i_j+1}}}$ so that $I(w_j,i_j)$ is good but its percolation lasts for time $\ell^{w_{ij}}$ and so that the size of $|\cup_j [i_j, i_j+\ell^{w_{ij}}]| \geq \frac{t}{4K} - t\epsilon$.

Choosing $\epsilon = \epsilon(K)$ sufficiently small, by Vitali Covering Lemma we can find $i_{1'}, i_{2'}, \cdots i_{f'}$ so that

(i) $\forall j' \neq j''\ \ [i_{j'}, i_{j'}+\ell^{w_{ij'}}] \cap [i_{j'}, i_{j''}+\ell^{w_{i,j''}}] = \emptyset$

(ii) $|\cup_j [i_j, i_j+\ell^{x_{ij}}]| \geq \frac{t}{15K}$.

vitalliprob **Lemma 2.14.** *For fixed sequence $\{y_i\}\{v_j\}$ as above the probability of $i_{1'}, i_{2'}, \cdots, i_{f'}$ giving such intervals is at most $\Pi_{j=1}^f C_4 e^{-\frac{\tilde{\theta}t}{5K}} \leq C_4^{\frac{t}{4K}} e^{-\frac{\tilde{\theta}t}{5K}}$ where $\tilde{\theta} = \tilde{\theta}(q) \uparrow \infty$ as $q \to 1$.*

This is simply Proposition 2.4 and independence which can be used since our assumption is that the initial renormalized intervals for each interval $[i_{j'}, i_{j'}+\ell^{y_{ij'}}]$ is good.

Proof of Proposition 2.10. We first consider a fixed coding $\{y_i\}\{v_j\}$. We must count (or bound) the number of Vitali coverings $[i_{j'}, i_{j'}+l^{w_{ij'}}]$ yielding cardinality $t/(15K)$. To do this we must choose the sequence of (w_j, i_j). This number is less than the number of ways of choosing a subset from $m \leq t/2K$ distinct object, that is bounded by $2^{t/2K}$. This factor is to be multiplied by the number of ways of choosing the lengths $\ell^{w_{i,j'}}$. Again the number of these choices is bounded by the number of ways of choosing a subset of size equal to the cardinal of set $\{i_{j'}\}$ from $m \leq t/2K$ objects and once more is bounded by $2^{t/2K}$.

Thus from Lemmas 2.13 and 2.14, the probability that some coding of lentgh smaller than Nt fails to touch a good interval is bounded by (using Lemmas 2.13 and 2.14)

$$4^{\frac{t}{2K}}\left(e^{-\theta t} \ + \ C_4^{\frac{t}{4K}} e^{-\frac{\tilde{\theta} t}{5K}}\right).$$

Hence (using Lemma 2.11 and Lemma 2.12) the probability that there exists a dual path not meeting a percolating interval is bounded by

$$e^{-t} + 4^{\frac{t}{2K}}(C_1C_2)^t(e^{-\theta t} \ + \ C_4^{\frac{t}{4K}} e^{-\frac{\tilde{\theta} t}{5K}})$$

and the result follows. □

2.4. **Proof of Proposition 1.4.**

Fix a site $x \in V$ such that $d(x, z_0) \leq Rt$ for some fixed suitable $R > 0$ sufficiently large. By Proposition 2.10, outside an event of exponential small probability, every path X in $\mathcal{D}(x, t, 3t/4)$ touchs at some point $(y, s) \in V \times [3t/4, t]$ a K normalized "dual" contact process that survives until time $t/4$. Since a dual path X in $\mathcal{D}(x, t, 3t/4)$ gets to a distance of Rt from x with exponential small probability, we can suppose that $d(y, z_0) \leq 2Rt$. By Proposition 2.9, the number of occupied sites among $\{z_0, ..., z_{Rt}\}$ of the K normalized dual contact process at time $t/4$ is at least $7R/10$ with probability $1 - e^{-ct}$. By Proposition 2.3 at least $R/2$ sites among the same $\{z_0, ..., z_{Rt}\}$ are occupied for both processes η and $\tilde{\eta}$ at time $t/4$ with probability $1 - e^{-ct}$. Therefore, outside an event of exponentially small probability there exists z_j such that $\eta(z_j) = \tilde{\eta}(z_j) = 1$ and this one is carried by a navigating path to y at time s, i.e, we also have $\eta(s) = \tilde{\eta}(s) = 1$, thus X is t-activated. We finish summing over all possible paths in $\mathcal{D}(x, t, 3t/4)$ analogously to what we did in Section 2.3. By an appropriate choice of the constants, we obtain Proposition 1.4.

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