SPECTRUM OF STOCHASTIC ADDING MACHINES AND FIBERED JULIA SETS

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Abstract. Consider the basic algorithm to perform the transformation $n \mapsto n+1$, changing digits of the d-adic expansion of n one by one. We obtain a family of Markov chains on the non-negative integers through sucessive and independent applications of the algorithm modified by a parametrized stochastic rule that randomly prevents one of the steps in the algorithm to finish. The object of study in this paper are the spectra of the transition operators of these Markov chains. The spectra of these Markov chains turn out to be fibered Julia sets of fibered polynomials. This enable us to analyze their topological and analytical properties with respect to the underlying parameters of the Markov chains.

1. Introduction

Binary representations of real numbers have many useful applications in science. One cares not only on how transformations on sets of real numbers can be described through their binary representations but also on how these transformations can be performed algorithmically. The transformation that associate to a natural number its successor, adding one to the number, is one of the simplest to be described by binary representations. A basic algorithm to perform the transformation $n \mapsto n+1$, changing binary digits one by one, requires less than $\lfloor \log_2(n) \rfloor + 1$ steps. Killeen and Taylor [6] proposed a stochastic rule that randomly prevents one of the steps in the algorithm to finish resulting in a number smaller than $n+1$. Successive interactions of Killeen and Taylor rule give rise to a Markov chain on $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$ whose transition operator has important spectral properties. In particular, the spectrum of the transition operator in l^∞ is equal to the filled-in Julia set of a quadratic map. Here we propose a generalization of Killeen and Taylor machine as well as of the results obtained in [2] and [6].

We extend these results in two directions. On the one hand, we consider not only binary representation but also d-adic extension of the natural

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numbers. On the other hand, the stochastic rule we consider is more general : the iteration of the adding algorithm is randomized through Bernoulli variables whose parameters change at each step (see below).

Let us fix a positive integer $d \geq 2$. Set

$$\Gamma = \Gamma_d := \left\{ (a_j)_{j=1}^{+\infty} \in \{0, ..., d-1\}^{\mathbb{N}} : \sum_{j=1}^{+\infty} a_j < \infty \right\}.$$

There is a one to one map from $\mathbb{Z}_+$ to Γ that associates to each n a sequence $(a_j(n))_{j=1}^{+\infty}$ such that

$$n = \sum_{j=1}^{+\infty} a_j(n) d^{j-1}.$$

The right hand side of the previous equality is called the d-adic expansion of n and $a_j(n)$ is called the jth digit of the expansion. The map $n \mapsto n+1$ operates on Γ in the following way: we define the counter $\zeta_n = \zeta_{d,n} := \min\{j \geq 1 : a_j(n) \neq d-1\}$ then

$$a_j(n+1) = \begin{cases} 0 & , \ j < \zeta_n \\ a_j(n)+1 & , \ j = \zeta_n \\ a_j(n) & , \ j > \zeta_n . \end{cases}$$

So an adding machine algorithm, that maps n to $n+1$ using d-adic expansions by changing one digit on each step, is performed in ζ_n steps in the following way: the first $\zeta_n - 1$ digits are replaced by zero recursively and in $\zeta_n th$ step we add one to the $\zeta_n th$ digit (basically we are adding one modulus d on each step). Note that $0 \leq \zeta_n \leq \lfloor \log_d(n) \rfloor + 1$.

As an example consider $d = 3$ and $n = 98 = 2 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^4$, then the adding machine algorithm is performed in $\zeta_{3,98} = 3$ steps as follows:

$$22101 \mapsto 02101 \mapsto 00101 \mapsto 00201. \tag{1.1}$$

Now suppose that for each step of the adding machine algorithm, independently of any other step, there is a positive probability that the information about the counter get lost, thus making the algorithm to stop. This implies that the outcome of the adding machine is a random variable. We call this procedure the adding machine algorithm with fallible counter, or simply $\mathrm{AMFC_d}$ where d represents the base.

Formally, we fix a sequence $(p_j)_{j=1}^{+\infty}$ of real numbers in $(0,1]$ and a sequence $(\xi_j)_{j=1}^{+\infty}$ of independent random variables such that ξ_j is a Bernoulli distribution with parameter p_j. Define the random time $\tau = \inf\{j : \xi_j = 0\}$. Then the $\mathrm{AMFC_d}$ is defined by applying the adding machine algorithm to n and stopping at the step τ if $\tau \leq \zeta_n$ (this means that steps $j \geq \tau$ are not performed).

Let us return to the previous example with $d = 3$ and $n = 98$. If $\xi_1 = 1$ and $\xi_2 = 0$, which occurs with probability $p_1(1-p_2)$ then $\tau = 2 < 3 = \zeta_{98}$. Thus the $\mathrm{AMFC_3}$ applied to $n = 98$ stops just after step one giving an outcome of $2\cdot 3+1\cdot 3^2+1\cdot 3^4=96$, see scheme in 1.1. Indeed the probability distribution of the outcome of the $\mathrm{AMFC_3}$ applied to $n=98$ is

$$22101\,(98) \mapsto \begin{cases} 22101\,(98) & \text{, with probability } 1-p_1 \\ 02101\,(96) & \text{, with probability } p_1(1-p_2) \\ 00101\,(90) & \text{, with probability } p_1p_2(1-p_3) \\ 00201\,(99) & \text{, with probability } p_1p_2p_3\,. \end{cases}$$

Now fix a initial, possibly random, state $X(0)\in\mathbb{Z}_+$. We apply recursively the $\mathrm{AMFC_d}$ to its successive outcomes starting at $X(0)$ and using independent sequences of Bernoulli random variables at different times. These random sequences are associated to the same fixed sequence of probabilities $(p_j)_{j=1}^{+\infty}$. In this way, we generate a discrete time-homogeneous Markov chain $(X(t))_{t\geq 0}$. This Markov chain is irreducible if and only if $p_j<1$ for infinitely many j's. During the rest of the paper we will assume that the previous condition is satisfied and the chain is irreducible.

As mentioned before, Killeen and Taylor [6], considered this stochastic machine in the case $d=2$ and $p_n=p$ for all $n\in\mathbb{N}$. They proved that the spectrum of S_2 in l^∞ is equal to the filled-in Julia set of a quadratic map. In [2], El Abdalaoui and Messaoudi, studied the spectrum of S_2 acting in other Banach spaces as C_0 and $l^\alpha(\mathbb{Z}_+),\ \alpha\geq 1$. Messaoudi and Smania [7] also defined the stochastic adding machine in the case where the base of numeration is not constant. In particular, they considered the case where the base is the Fibonacci sequence. They proved that the eigenvalues of the spectrum of the transition operator acting in $l^\infty(\mathbb{Z}_+)$ is connected to the Julia set of an endomorphism of $\mathbb{C}^2$ (see also [9] for the case where the base belong to a class of a recurrent sequences of degree 2).

In this paper, we study convergence and spectral properties of the $\mathrm{AMFC_d}$ Markov chain. We prove that the $\mathrm{AMFC_d}$ Markov Chain is null recurrent if and only if $\prod_{j=1}^{+\infty}p_j=0$. Otherwise the chain is transient. We also prove that the spectrum of the transition operator S_d acting on l^∞ is equal to the filled-in fibered Julia set E defined by

$$E=E_d:=\left\{z\in\mathbb{C}:\limsup_{j\to+\infty}|\tilde{f}_j(z)|<+\infty\right\}.$$

where $\tilde{f}_j:=f_j\circ\ldots\circ f_1$ for all $j\geq 1$ and $f_j:\mathbb{C}\to\mathbb{C}$, is the function defined by

$$f_j(z):=\left(\frac{z-(1-p_j)}{p_j}\right)^d.$$

We shall study topological properties of the filled-in fibered Julia set E. In particular, we give sufficient conditions on the sequence $(p_n)_{n\geq 1}$ to ensure that E is a connected set or has a finite number of connected components, or is a Cantor set.

We also study some properties of the fibered Julia set ∂E, in particular, we introduce the Green function of ∂E and prove that there exists $0<\rho<1$ and $\kappa>1$such that whenever $p_i\in[\rho,1]$ for all $i\geq 2$, then ∂E is a κ-quasicircle.

The paper is organized as follows: In section 2 we obtain the transition operator of the $\mathrm{AMFC_d}$ Markov chain and we give a necessary and sufficient condition for recurrence and transience; Section 3 is devoted to provide an exact description of the spectra of these transition operators acting on l^∞; Section 4 contains results about connectedness properties of the filled-in fibered Julia sets E; In section 5, further properties of ∂E are established in connection with properties of the associated fibered polynomials.

2. Transition operators and recurrence of $\mathrm{AMFC_d}$ chains

In this section $(X(t))_{t\geq 0}$ is an irreducible $\mathrm{AMFC_d}$ Markov chain associated to a sequence of probabilities $\bar{p} = (p_j)_{j=1}^{+\infty}$. Our first aim is to describe the transition probabilities of $(X(t))_{t\geq 0}$ which we denote $s(n,m) = s_{\bar{p},d}(n,m) := P(X(t+1) = m | X(t) = n)$. They can be obtained directly from description of the chain. For every $n \geq 0$, recall the definition of the counter ζ_n, one has

$$s(n,m) = \begin{cases} (1-p_{r+1})\prod_{j=1}^{r} p_j & \text{if } \mathrm{m = n - \sum_{j=1}^{r}(d-1)d^{j-1}}\,, \\ & \quad r \leq \zeta_n - 1\,,\ \zeta_n \geq 2\,, \\ 1 - p_1 & \text{if } \mathrm{m = n}\,, \\ \prod_{j=1}^{\zeta_n} p_j & \text{if } \mathrm{m = n+1}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.1}$$

From the exact expressions above, the transitions probabilities satisfy a property of self-similarity. Indeed, the following Lemma is straightforward

Lemma 2.1. *For all $j \geq 2$ and for all $d^{j-1} \leq n \leq d^j - 2$,*

$$s(n,m) = \begin{cases} s(n - a_j(n)d^{j-1}, m - a_j(n)d^{j-1}) & ,\ d^{j-1} \leq m \leq d^j - 1\,, \\ 0 & ,\ \textit{otherwise.} \end{cases}$$

Note that $\zeta_n \leq j$ for this choice of n.
Moreover, if $n = d^j - 1$, we have $\zeta_n = j+1$ thus $s(d^j - 1, d^j) = \prod_{l=1}^{j+1} p_l$ and

$$s(d^j - 1, d^j - d^r) = (1 - p_{r+1}) \prod_{l=1}^{r} p_l\,,\ 1 \leq r \leq j\,. \tag{2.2}$$

With the transition probabilities, we obtain the countable transition matrix of the $\mathrm{AMFC_d}$ Markov chain $S = S_d = [s(n,m)]_{n,m\geq 0}$. To help the reader, the first entries of the matrix S_2 are given below:

$$\begin{bmatrix} 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p_1(1-p_2) & 1-p_1 & p_1p_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p_1p_2(1-p_3) & 0 & p_1(1-p_2) & 1-p_1 & p_1p_2p_3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & p_1(1-p_2) & 1-p_1 & p_1p_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & \cdots \\ p_1p_2p_3(1-p_4) & 0 & 0 & 0 & p_1p_2(1-p_3) & 0 & p_1(1-p_2) & 1-p_1 & p_1p_2p_3p_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For S_3, the first entries of the matrix are given below:

$$\begin{bmatrix} 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & \cdots \\ p_1p_2(1-p_3) & 0 & 0 & 0 & 0 & 0 & p_1(1-p_2) & 0 & 1-p_1 & p_1p_2p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The transition operator induced by S, acting on $l^\infty(\mathbb{Z}_+)$, will also be denoted by S. From a result in [2] we see that its restriction on $l^\alpha(\mathbb{Z}_+)$, $\alpha \geq 1$, is a well defined operator on each of these spaces. Note that S is doubly stochastic if and only if $\prod_{j=1}^{+\infty} p_j = 0$. In fact S is stochastic and the sum of coefficients of every column is 1, except the first one whose sum is $1 - \prod_{j=1}^{+\infty} p_j$.

In the next Proposition, we obtain a necessary and sufficient condition for recurrence of the $\mathrm{AMFC_d}$ Markov chain.

Proposition 2.2. *The* $\mathrm{AMFC_d}$ *Markov chain is null recurrent if and only if*

$$\prod_{j=1}^{+\infty} p_j = 0\,. \tag{2.3}$$

Otherwise the chain is transient.

Proof. We start showing that condition (2.3) is necessary and sufficient to guarantee the recurrence of the $\mathrm{AMFC_d}$ Markov chain. From classical Markov chain Theory, the $\mathrm{AMFC_d}$ Markov chain is transient if and only if there exists a sequence $v = (v_j)_{j=1}^{+\infty}$ such that $0 < v_j \leq 1$ and

$$v_j = \sum_{m=1}^{+\infty} s(j,m)\, v_m\,,\ j \geq 1\,, \tag{2.4}$$

i.e, $\tilde{S}_d\, v = v$ where $\tilde{S}_d$ is obtained from S removing its first line and column.

Suppose that $v = (v_j)_{j=1}^{+\infty}$ satisfies the above conditions. We claim that

$$v_{d^l+j} = v_{d^l}\,,\ \text{ for every } \ l \geq 0 \ \text{ and } \ j \in \{1, ..., (d-1)d^l - 1\}\,. \tag{2.5}$$

The proof follows by induction. Indeed, for $j \in \{1, ..., (d-1)d^l - 1\}$, suppose that $v_{d^l} = v_{d^l+r}$, for all $0 \leq r \leq j-1$ we have that $v_{d^l+j-1} = \sum_{m=1}^{j} s(d^l + j - 1, m)\, v_m$. Since $s(d^l + j - 1, m) = 0$ for all $0 \leq m < d^l$, we have

$$\begin{aligned} v_{d^l+j-1} &= \sum_{r=0}^{j} s(d^l + j - 1, d^l + r)\, v_{d^l+r} \\ &= \left(\sum_{r=0}^{j} s(d^l + j - 1, d^l + r) \right) v_{d^l} \\ &\qquad + s(d^l + j - 1, d^l + j)(v_{d^l+j} - v_{d^l})\,. \end{aligned} \tag{2.6}$$

Using the fact that $j \in \{1, ..., (d-1)d^l - 1\}$ note that

$$\sum_{r=0}^{j} s(d^l + j - 1, d^l + r) = 1\,,$$

thus, since $s(d^l+j-1, d^l+j) > 0$, from (2.6), we have that $v_{d^l+j} = v_{d^l+j-1} = v_{d^l}$. This proves the claim.

It remains to obtain $v_{d^{l+1}}$ from v_{d^l} for $l \geq 0$. First note that (2.5) implies $v_{d^{l+1}-d^r} = v_{d^l}$ for $0 \leq r \leq l$. From the transition probabilities expression in (2.2), if we let $p_0 = 1$, we have that

$$\begin{aligned} v_{d^l} &= v_{d^{l+1}-1} = (p_0 ... p_{l+2}) v_{d^{l+1}} + \sum_{r=0}^{l} (p_0 ... p_r - p_0 ... p_{r+1}) v_{d^{l+1}-d^r}\,. \\ &= (p_0 ... p_{l+2}) v_{d^{l+1}} + (1 - p_0 ... p_{l+1}) v_{d^l}\,. \end{aligned}$$

Therefore for every $l \geq 1$

$$v_{d^l} = \frac{v_{d^{l-1}}}{p_{l+1}} = \frac{v_1}{\prod_{j=2}^{l+1} p_j}\,.$$

From this equality, we get to the conclusion that v exists and the chain is transient if and only if

$$\prod_{j=1}^{+\infty} p_j > 0\,.$$

Now suppose that we are in the recurrent case. Since S is a irreducible countable doubly stochastic matrix, it is simple to verify that the $\mathrm{AMFC_d}$ have no finite invariant measure and then cannot be positive recurrent. □

We can also obtain the recurrence/transience condition of Proposition 2.2 through probabilistic arguments. We just describe roughly these arguments leaving the details to the reader. Let $(X(t))_{t\geq 0}$ be the $\mathrm{AMFC_d}$ Markov chain starting at $X(0) = 0$. For $n \geq 0$, denote the first hitting time of state d^n by

$$\tau_n = \min\{t \geq 1 : X(t) = d^n\}$$

and the number of visits to 0 before the random time τ_n by N_n. Then the expectation of N_n is

$$\Big(\prod_{j=1}^{n+1} p_j\Big)^{-1}\,. \tag{2.7}$$

We check this by induction. If $n = 0$ then $N_0 = \tau_0$. But τ_0 is a geometric random variable with parameter p_1, which has expectation p_1^{-1}. Now suppose that (2.7) holds for n. When the chain reachs state d^n, it must get to $d^{n+1} - 1$ to attempt a return to 0. If it returns to 0, it must spend a time with the same distribution of τ_n to reach d^n again. Moreover, the number of visits to $d^{n+1} - 1$ resulting in a jump to 0 or a jump to d^{n+1} is a geometric random variable of parameter p_{n+2}^{-1}. By the Markov property, the expectation of N_{n+1} is the expectation of N_n times p_{n+2}^{-1}. This means that (2.7) also holds for $n+1$.

Letting n goes to infinity, we get that the expected number of visits to 0 is $\left(\prod_{j=1}^{\infty} p_j\right)^{-1}$. Moreover, from classical Markov chain theory, the probability that the chain never returns to 0 is the inverse of the expected number of visits to 0, which is $\prod_{j=1}^{\infty} p_j$. Therefore, the chain is recurrent if and only if $\prod_{j=1}^{\infty} p_j = 0$.

Finally, let us just point out that the previous argument also allows us to show that the expectation of τ_n is given by $d^n\left(\prod_{j=1}^{n+1} p_j\right)^{-1}$. Indeed, the expected number of visits to each one of the states $0, ..., d^n - 1$ before time τ_n is $\left(\prod_{j=1}^{n+1} p_j\right)^{-1}$.

3. Spectra of transition operators of $\mathrm{AMFC_d}$ chains

In this section we describe the spectrum of the transition operator of an $\mathrm{AMFC_d}$ acting on $l^\infty(\mathbb{Z}_+)$ for a fixed sequence $\bar{p} = (p_j)_{j=1}^{+\infty}$ with p_j in $(0, 1]$ satisfying the irreducibility condition. We start introducing some notation. Let $f_j : \mathbb{C} \to \mathbb{C}$, $j \geq 1$, be the function defined by

$$f_j(z) := \left(\frac{z - (1 - p_j)}{p_j}\right)^d .$$

Also let $\tilde{f}_0 := Id$, $\tilde{f}_j := f_j \circ ... \circ f_1$ and

$$E_{\bar{p}} = E := \left\{z \in \mathbb{C} : \limsup_{j\to+\infty} |\tilde{f}_j(z)| < +\infty\right\}.$$

We use the notation $\mathbb{D}(w, r) = \{z \in \mathbb{C} : |w - z| < r\}$ and $\overline{\mathbb{D}(w, r)} = \{z \in \mathbb{C} : |w - z| \leq r\}$.

Lemma 3.1. *The set $E_{\bar{p}}$ is included in the closed disk $\overline{\mathbb{D}(1 - p_1, p_1)}$. Moreover, for all $z \in E_{\bar{p}}$ and $j \geq 1$, $\tilde{f}_j(z)$ belongs to the disk $\overline{\mathbb{D}(1 - p_{j+1}, p_{j+1})}$.*

Proof. Take $p_j \in (0, 1)$ and $z \in \mathbb{C}$ with $|z| > 1$ then

$$\left|\frac{z - (1 - p_j)}{p_j}\right| \geq \frac{|z| - (1 - p_j)}{p_j} = \frac{|z| - 1}{p_j} + 1 > |z| > 1 .$$

Thus, we obtain, for every $z \in \mathbb{C}$ with $|z| > 1$ and $j \geq 1$, that

$$|f_j(z)| > |z|^d > 1 .$$

Now suppose $|\tilde{f}_r(z)| > 1$ for some $r > 1$, then by induction one can show that for $j > r$

$$|\tilde{f}_j(z)| \geq |\tilde{f}_r(z)|^{d^{j-r}} . \tag{3.1}$$

From (3.1) we see that $\lim_{j\to+\infty} |\tilde{f}_j(z)| = +\infty$ whenever $|\tilde{f}_r(z)| > 1$ for some $r > 1$. In particular, if $|\tilde{f}_r(z)| > 1$ then $z \notin E$.

As regard the second part of the Lemma, suppose $|z - (1 - p_1)| > p_1$, this implies that $|f_1(z)| > 1$ and then $z \notin E$. Analogously, if $|\tilde{f}_j(z) - (1-p_{j+1})| > p_{j+1}$, we have that $|\tilde{f}_{j+1}(z)| > 1$ and then $z \notin E$. □

Corollary 3.2. *We have the following equality*

$$\begin{aligned} E_{\bar{p}} &= \overline{\mathbb{D}(1-p_1,p_1)} \cap \bigcap_{j=1}^{\infty} \tilde{f}_j^{-1}\big(\overline{\mathbb{D}(1-p_{j+1},p_{j+1})}\big) \\ &= \overline{\mathbb{D}(0,1)} \cap \bigcap_{j=1}^{\infty} \tilde{f}_j^{-1}\big(\overline{\mathbb{D}(0,1)}\big) . \end{aligned}$$

Proposition 3.3. *The point spectrum of S in $l^\infty(\mathbb{Z}_+)$ is equal to E. Furthermore, fix $\lambda \in E$ and $v_0 > 0$ and define*

$$v_n = v_0 \prod_{r=1}^{\lfloor \log_d(n) \rfloor + 1} (q_\lambda(r))^{a_r(n)} \, , \; n \geq 0 \tag{3.2}$$

where $a_r(n)$ is the rth digit of n in its d-adic expansion and

$$q_\lambda(r) = (\, h_r \circ \tilde{f}_{r-1} \,)(\lambda) \tag{3.3}$$

with

$$h_r(z) = \frac{z}{p_r} - \frac{1-p_r}{p_r}. \tag{3.4}$$

Thus $(v_n)_{n=0}^{+\infty}$ is, up to multiplication by a constant, the unique right eigenvector of S in $l^\infty(\mathbb{N})$ with eigenvalue λ.

Proof. Since S is stochastic, its spectrum is a subset of the closed disk $\overline{\mathbb{D}(0,1)}$. Let us fix $\lambda \in \overline{\mathbb{D}(0,1)}$ and $(v_n)_{n=0}^{+\infty}$ the sequence defined in the statement. Then the proof of the Proposition follows from the two claims below:

Claim 1: $(|q_\lambda(j)|)_{j=1}^{+\infty}$ is bounded above by one if and only if $\lambda \in E$, otherwise it is unbounded. In particular, $(v_n)_{n=0}^{+\infty}$ is a well defined element of $l^\infty(\mathbb{Z}_+)$ if and only if $\lambda \in E$.

Claim 2: If $\lambda \in E$, $(v_n)_{n=0}^{+\infty}$ is, up to multiplication by a constant, the unique right eigenvector of S with eigenvalue λ.

Proof of Claim 1: Simple to verify from definition (3.2), Lemma 3.1 and the fact that

$$\inf_{j\geq 1} |h_j(x)| \geq |x| \, .$$

Proof of Claim 2: Let $v = (v_n)_{n\geq 0}$ be a sequence of real numbers and suppose that $(S\,v)_n = \lambda\, v_n$ for every $n \geq 0$. We shall prove that v satisfies

(3.2). The proof is based on the following representation

$$(S\,v)_n \;=\; \left(\prod_{j=1}^{\zeta_n} p_j\right) v_{n+1} + (1-p_1)v_n + \sum_{r=1}^{\zeta_n-1}\left(\prod_{j=1}^{r} p_j\right)(1-p_{r+1})v_{n-\sum_{j=1}^{r}(d-1)d^{j-1}}\,, \qquad (3.5)$$

for $\zeta_n \geq 2$ and $(S\,v)_n = p_1 v_{n+1} + (1-p_1)v_n$ if $\zeta_n = 1$. This representation follows directly from the definition of the transition probabilities in (2.1). From (3.5), we show (3.2) by induction.

Indeed, for $n = 1$ we have that

$$\lambda v_0 = (1-p_1)v_0 + p_1 v_1 \;\Rightarrow\; v_1 = \left(\frac{\lambda-(1-p_1)}{p_1}\right) v_0 = q_\lambda(1)\, v_0\,.$$

Now fix $n \geq 1$ and suppose that (3.2) holds for every $1 \leq j \leq n$. By (3.5), since $(S\,v)_n = \lambda\, v_n$, we have that

$$\frac{v_{n+1}}{v_0\,\prod_{r=\zeta_n+1}^{\lfloor \log_d(n)\rfloor+1}(q_\lambda(r))^{a_r(n)}} \qquad (3.6)$$

is equal to

$$\frac{[\lambda-(1-p_1)]\left[\prod_{r=1}^{\zeta_n-1}(q_\lambda(r))^{d-1}\right](q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}}{\prod_{j=1}^{\zeta_n}p_j} + \frac{(1-p_2)\left[\prod_{r=2}^{\zeta_n-1}(q_\lambda(r))^{d-1}\right](q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}}{\prod_{j=2}^{\zeta_n}p_j} \;...\; + \frac{1-p_{\zeta_n}}{p_{\zeta_n}}(q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}\,. \qquad (3.7)$$

Since

$$q_\lambda(1) = \frac{\lambda-(1-p_1)}{p_1}\,,$$

the first term in (3.7) is equal to

$$\frac{q_\lambda(1)^d\left[\prod_{r=2}^{\zeta_n-1}(q_\lambda(r))^{d-1}\right](q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}}{\prod_{j=2}^{\zeta_n}p_j}\,.$$

Summing with the second term we get

$$\left(\frac{(q_\lambda(1))^d-(1-p_2)}{p_2}\right)\frac{\left[\prod_{r=2}^{\zeta_n-1}(q_\lambda(r))^{d-1}\right](q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}}{\prod_{j=3}^{\zeta_n}p_j}\,,$$

which is equal to

$$\frac{(q_\lambda(2))^d \left[\prod_{r=3}^{\zeta_n-1}(q_\lambda(r))^{d-1}\right](q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}}{\prod_{j=3}^{\zeta_n} p_j}.$$

By induction we have that the sum of the first $\zeta_{d.n}-1$ terms in (3.7) is equal to

$$\frac{(q_\lambda(\zeta_n-1))^d\,(q_\lambda(\zeta_n))^{a_{\zeta_n}(n)}}{p_{\zeta_n}}.$$

Finally, summing the previous expression with the last term in (3.7) we have that (3.6) is equal to

$$\frac{(q_\lambda(\zeta_n-1))^d-(1-p_{\zeta_n})}{p_{\zeta_n}}\,(q_\lambda(\zeta_n))^{a_{\zeta_n}(n)} = (q_\lambda(\zeta_n))^{a_{\zeta_n}(n)+1},$$

Therefore,

$$\begin{aligned} v_{n+1} &= v_0\,(q_\lambda(\zeta_n))^{a_{\zeta_n}(n)+1}\prod_{r=\zeta_n+1}^{\lfloor\log_d(n)\rfloor+1}(q_\lambda(r))^{a_r(n)} \\ &= v_0\prod_{r=1}^{\lfloor\log_d(n+1)\rfloor+1}(q_\lambda(r))^{a_r(n+1)}, \end{aligned}$$

which, by induction, completes the proof of Claim 2. □

Denote by $\sigma_{\bar p}$ the spectrum of $S_{\bar p}$ in $l^\infty(\mathbb{Z}_+)$. We have proved in the previous proposition that $\sigma_{\bar p} \supset E_{\bar p} = E$. In the next Proposition, we show that $\sigma_{\bar p} \subset E_{\bar p}$.

Theorem 3.4. *The spectrum of $S_{\bar p}$ is equal to E.*

Proof. We prove here that $\sigma_{\bar p} \subset E$. Let us denote by $\tau : \mathbb{Z}_+ \to \mathbb{Z}_+$ the shift map $\tau(n) = n+1$ and $\bar p_n := (p_{n+j})_{j=0}^\infty$ for a given $n \in \mathbb{N}$.

Denote $\tilde S_{\bar p}$ the operator

$$\tilde S_{\bar p} := \frac{S_{\bar p} - (1-p_1)Id}{p_1},$$

which is also a stochastic operator acting on $\mathbb{Z}_+$. It is associated to a irreducible Markov chain with period d. Thus $\tilde S_{\bar p}^d$ has d communication classes.

It is straightforward to verify that the communication classes of $\tilde S_{\bar p}^d$ are

$$\{\, j \in \mathbb{N} \;:\; j \equiv n \mod d\},\quad 0 \le n \le d-1.$$

Furthermore, $\tilde S_{\bar p}^d$ acts on each of these classes as a copy of $S_{\bar p_2}$. Therefore, the spectrum of $\tilde S_{\bar p}^d$ is equal to the spectrum of $S_{\bar p_2}$. Since, $\tilde S_{\bar p}^d = \tilde f_1(S_{\bar p})$, by the Spectral Mapping Theorem([4]), we have that

$$\tilde f_1(\sigma_{\bar p}) = \sigma_{\bar p_2}.$$

By induction, we have that

$$\tilde{f}_{j+1}(\sigma_{\bar{p}}) = \sigma_{\bar{p}_{j+1}} \,,$$

for every $j \geq 1$. Since, $S_{\bar{p}_{j+1}}$ is a stochastic operator, its spectrum is a subset of $D(0,1)$. Therefore

$$|\tilde{f}_{j+1}(\lambda)| \leq 1 \,,$$

for every j and $\lambda \in \sigma_{\bar{p}}$. This implies that $\sigma_{\bar{p}} \subset E$. □

Remark 3.1. *Let $(X(t))_{t\geq 0}$ be an irreducible recurrent* $\mathrm{AMFC_d}$ *Markov chain starting at* 0. *Suppose that $\lambda \in \mathbb{R} \cap E$, $\lambda \neq 1$ and $v = (v_n)_{n\geq 0}$ is an eigenvector associated to λ with $v_0 > 0$. Since $E \subset D(0,1)$, then v is a super-harmonic function on $l^\infty(\mathbb{Z}_+)$ with respect to S, i.e., $(Sv)_n \leq v_n$, for all $n \geq 0$. Therefore, by (3.2) and from the Potential Theory for Markov chains, we have that $(v_{X(t)})_{t\in\mathbb{Z}_+}$ is a bounded supermartingale. If $(v_{X(t)})_{t\in\mathbb{Z}_+}$ is positive, by the convergence theorem for supermartingales, we have that there exists a bounded $\mathbb{Z}_+$ valued random variable v_∞ such that $\lim_{t\to\infty} v_{X(t)} = v_\infty$ almost surely. Since the chain is irreducible and recurrent, this can happen only if v is constant. But for $\lambda \neq 1$, by (3.2), v is non-constant. Therefore, either $\mathbb{R} \cap E = \{1\}$ or $v_n < 0$ for some n. However we have $v_n < 0$ for some n if and only if $\tilde{f}_j(\lambda) < 0$ for some $j \geq 1$, which, for a constant sequence $(p_i)_{i\geq 1}$, implies that $(-1,0) \cap E \neq \emptyset$.*

4. Connectedness properties of the spectra

This section is devoted to the study of the connectedness of E and its complement. According to the parity of d, we obtain conditions for E to be connected, have a finite number of connected components or to be a Cantor set.

Proposition 4.1. *The set $\mathbb{C} \setminus E$ is connected.*

Proof. By Lemma 3.1, we have $E = \bigcap_{n=1}^{+\infty} \tilde{f}_n^{-1}\big(\overline{\mathbb{D}(0,1)}\big)$ where $\tilde{f}_{n+1}^{-1}\big(\overline{\mathbb{D}(0,1)}\big) \subset \tilde{f}_n^{-1}\big(\overline{\mathbb{D}(0,1)}\big)$ for every integer $n \geq 1$. Then $\mathbb{C} \setminus E = \bigcup_{n=1}^{+\infty} \mathbb{C} \setminus \tilde{f}_n^{-1}\big(\overline{\mathbb{D}(0,1)}\big)$. Since for any integer n, f_n is a polynomial function, then $X_n = \mathbb{C} \setminus \tilde{f}_n^{-1}\big(\overline{\mathbb{D}(0,1)}\big)$ is a connected set. Since X_n is an increasing sequence, we deduce that $\mathbb{C} \setminus E$ is also a connected set. □

In the rest of this this paper we need to introduce some more notations. Let $g_j : \mathbb{C} \to \mathbb{C}$, $j \geq 2$, be the function defined by

$$g_j(z) := \frac{1}{p_j} z^d - \frac{(1-p_j)}{p_j},$$

and $\tilde{g}_j := g_{j+1} \circ \ldots \circ g_2$. Let

$$K = \Big\{ z \in \mathbb{C} : \limsup_{j\to+\infty} |\tilde{g}_j(z)| < +\infty \Big\}. \tag{4.1}$$

Recall that $h_j(z) = \frac{z}{p_j} - \frac{1-p_j}{p_j}$, the functions g_{j+1} and f_j are conjugated in the following sense

$$g_{j+1} = h_{j+1} \circ f_j \circ h_j^{-1} \tag{4.2}$$

Since $\tilde{g}_j(h_1(z)) = h_{j+1} \circ \tilde{f}_j(z)$ for all $j \geq 2$, we deduce that

$$\text{if } \liminf_{i\to\infty} p_i > 0, \text{ then } E := h_1^{-1}(K) \tag{4.3}$$

According to Lemma 3.1, since h_1 maps the disk $\overline{\mathbb{D}(1-p_1, p_1)}$ to $\overline{\mathbb{D}(0,1)}$, we have that if $\liminf_{i\to\infty} p_i > 0$, then K is also included in the closed disk $\overline{\mathbb{D}(0,1)}$. Indeed, by the Lemma 3.1, we have an analogous of Corollary 3.2. In the case where $\liminf_{i\to\infty} p_i = 0$, the same results are true, and we have $h_1^{-1}(K) \subset E$.

Lemma 4.2. *Let $R > 1$ then $K = \bigcap_{n=2}^{+\infty} \tilde{g}_n^{-1}(\mathbb{D}(0,R))$ where $\tilde{g}_{n+1}^{-1}(\mathbb{D}(0,R)) \subset \tilde{g}_n^{-1}(\mathbb{D}(0,R))$ for every integer $n \geq 2$.*

Remark 4.1. ***1.*** *The use of $\tilde{g}_j$ instead of $\tilde{f}_j$ simplify the study of some topological properties of E. Indeed, since h_1 is a linear homeomorphism from $\mathbb{C}$ to $\mathbb{C}$, we have when $\liminf_{i\to\infty} p_i > 0$ that E and K are really the same up to a linear change of coordinates.*
2. *The analysis of the number of connected components of E relies on the Riemann-Hurwitz Formula, see* [8]. *It relates the number of connected components of E with the number of critical points (counted with multiplicity) of $\tilde{f}_n$ that do not belong to E. The critical points of $\tilde{f}_n$ are of the form $z \in \mathbb{C}$ such that $z = 1 - p_1$ or $\tilde{f}_k(z) = 1 - p_{k+1}$ for some integer $1 \leq k < n$.*

As it will become clear just ahead, the fact that 0 belongs or not to K is relevant in the study of the connectedness of both E and K.

4.1. **Case where $p_i = p$ for all $i \geq 2$.** Here g_j does not depend on j and will be denoted by

$$g(z) := \frac{1}{p} z^d - \frac{(1-p)}{p}, \ z \in \mathbb{C}.$$

Lemma 4.3. *Suppose that $p_i = p$ for all $i \geq 2$. If $0 \in K$ then K is a connected set, otherwise K is a Cantor set.*

Proof. The result is a consequence of Lemma 4.2, Riemann-Hurwitz Formula, and the fact that 0 is the unique critical point of all g_n, $n \geq 2$. □

Proposition 4.4. *Let $p \in (0,1)$ be a fixed real number and suppose that $p_j = p$ for all $j \geq 2$, then the following properties are satisfied.*

(i) *If d is even, then E is connected if and only if $p \geq \frac{1}{2}$, otherwise E is a Cantor set.*

(ii) *If d is odd, then E is connected, if and only if, $p \geq \vartheta_d = d\theta_d^{d-1}$ where $\theta_d \in (0,1)$ is the unique non-negative solution of*

$$d\,\theta^{d-1} + (d-1)\,\theta^d = 1\,, \tag{4.4}$$

otherwise E is a Cantor set.

Remark 4.2. ***1.*** *Since $\theta_d \in (0,1)$, we have that $d\,\theta^{d-1} > (d-1)\,\theta^d$ and by equation (4.4) we have that $\vartheta_d > 1/2$. This implies a noticeable difference with the case d even.*
2. *If $d = 3$ then $\theta_3 = 1/2$ and $\vartheta_3 = 3/4$.*
3. *We have that ϑ_d decreases to $1/2$ as $d \to \infty$.*

Proof. of Proposition 4.4.

(i) Assume that d is even. If $p < \frac{1}{2}$, then $g(0) = -\frac{(1-p)}{p} < -1$, hence $0 \notin K$. By Lemma 4.3 and Remark 4.1, K and E are Cantor sets. Now, suppose $p \geq \frac{1}{2}$ and let $-1 \leq x \leq 1$, then we see easily by induction on n that $-1 \leq -\frac{(1-p)}{p} \leq g^n(x) \leq 1$ for all $n \in \mathbb{N}$. Hence $0 \in K$. By Lemma 4.3 and Remark 4.1, K and E are connected.

(ii) Now, assume that d is odd .

Claim : $0 \in K$ if and only if the equation $x = g(x)$ has a real solution $-1 \leq x \leq 0$.

Proof of the Claim: Assume that there exists a real number $x \leq 0$ such that $x = g(x)$. Then, from monotonicity properties of g, for any integer $n \geq 0$, we have $x \leq g^n(0) \leq 0$, therefore $0 \in K$.

Now suppose that $0 \in K$ and put $l = \inf\{g^n(0) : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $g^n(0) = \frac{1}{p}g^{n-1}(0)^d - \frac{1-p}{p} < l+\varepsilon$, then $g(l) = \frac{l^d-(1-p)}{p} < l+\varepsilon$. On the other hand $l \leq g^{n+1}(0) < \frac{1}{p}(l+\varepsilon)^d - \frac{1-p}{p}$. Since ϵ is arbitrary, then $l = g(l) = \frac{l^d-(1-p)}{p}$. From the fact that $-1 \leq l \leq 0$, the claim is proved.

Now consider the equation $\psi(x) = p(g(x) - x) = x^d - px - (1-p)$. Since $\psi(-1) = 2(p-1) < 0$, $\psi(0) = p - 1 < 0$ and $\max_{x \leq 0} \psi(x) \leq \psi\Big(- (\frac{p}{d})^{\frac{1}{d-1}}\Big)$, we have that

$$0 \in K \Longleftrightarrow \psi\Big(- \Big(\frac{p}{d}\Big)^{\frac{1}{d-1}}\Big) \geq 0.$$

Consider the function $\phi(p) = \psi\big(- (\frac{p}{d})^{\frac{1}{d-1}}\big) = (d-1)\big((\frac{p}{d})^{\frac{d}{d-1}}\big) - (1-p)$. As for any integer $d \geq 2$, we have $\phi(0) < 0, \phi(1) > 0$ and ϕ is increasing, we deduce that there exists a unique real number $0 < \vartheta_d < 1$ such that $\phi(\vartheta_d) = \psi(-\Big(\frac{\vartheta_d}{d})^{\frac{1}{d-1}}\Big) = 0$. Since $\partial_p\psi(x) = -x + 1 \geq 0$ for all $x \leq 0$, we obtain that $0 \in E$ if and only if $p \geq \vartheta_d$. On the other hand, if $\theta_d = \Big(\frac{\vartheta_d}{d}\Big)^{\frac{1}{d-1}}$, then $(d-1)\theta_d^d + d\theta_d^{d-1} - 1 = 0$ and $\vartheta_d = d\theta_d^{d-1}$. $\square$

4.2. **Case where $(p_i)_{i\geq 2}$ is not constant.** Now we focus on the general setting, again there are two different behaviors with respect to the parity of d.

Proposition 4.5. *Assume that d is even and let $s = \#\{i \geq 2,\ p_i < \frac{1}{2}\}$. Then we have the following results*

(i) *If $s = 0$, then E is connected.*
(ii) *If $0 < s < +\infty$, then E has d^k connected components where $s \leq k \leq t-1$ where $p_t < \frac{1}{2}$ and $p_i \geq \frac{1}{2}$ for all $i > t$. In particular, if $p_i < 1/2$ for all $2 \leq i \leq n$ and $p_i \geq 1/2$ for all $i > n+1$, then E has exactly d^{n-1} connected components.*
(iii) *If $s = +\infty$, then E is a Cantor set.*

Proof. (i) Let $n \geq 2$ and z be a critical point of $\tilde{g}_n = g_{n+1} \circ \ldots \circ g_2$, then $z = 0$ or $\tilde{g}_i(z) = 0$ for some $2 \leq i \leq n$.

Suppose $s = 0$ then $p_i \geq \frac{1}{2}$ for all integer $i \geq 2$. It is easy to see that all integers $2 \leq k \leq m$, $-1 \leq 1 - \frac{1}{p_{m+1}} \leq \tilde{g}_m(0) \leq 1$. We deduce that all critical points of $\tilde{g}_n$ belong to K. Hence K is a connected set. Thus by (4.3), we deduce that E is connected,

(ii) Assume that $0 < s < +\infty$, then there exist s integers $k_1 < k_2 < \ldots < k_s$ such that $p_{k_j} < \frac{1}{2}$ for all $1 \leq j \leq s$. Hence $1 - \frac{1}{p_{k_j}} < -1$ for all $1 \leq j \leq s$.

Let $n > k_s$, $1 \leq j \leq s$, and z_j be a complex number such that $\tilde{g}_{k_j-2}(z_j) = 0$, then $\tilde{g}_n(z_j) = g_{n+1} \circ \ldots \circ g_{k_j+1}(1 - \frac{1}{p_{k_j}})$ diverges to $-\infty$ as $n \to \infty$. Hence $z_j \notin K$. Since z_j is a critical point of $\tilde{g}_n$. Using Riemann-Hurwitz formula, we deduce that K has at least d^s connected components. On the other hand, as $p_i \geq \frac{1}{2}$ for all $i > k_s$, we deduce that all complex number z satisfying that $\tilde{g}_m(z) = 0$, $m > k_s$, belongs to K, therefore E has at most d^{k_s} connected components.

(iii) Suppose that $s = +\infty$. If $\liminf_{i\to\infty} p_i > 0$, then by (4.3) and item (ii), we deduce that E is a Cantor set.

Now, suppose that $\liminf_{i\to\infty} p_i = 0$, Since if $R > 1$ is a real number, then $E = \bigcap_{n=1}^{+\infty} \tilde{f}_n^{-1}\mathbb{D}(0, R)$ where $\tilde{f}_{n+1}^{-1}\mathbb{D}(0, R) \subset \tilde{f}_n^{-1}\mathbb{D}(0, R)$ for every integer $n \geq 1$; and the fact that the critical points of $\tilde{f}_n$ are z such that $z = 1 - p_1$ or $\tilde{f}_k(z) = 1 - p_{k+1}$ for some integer $k < n$. Using the same idea done in the proof of item (ii), we deduce that E is a Cantor set. □

Proposition 4.6. *If d is odd, then the following assertions hold:*

(i) *E is connected if and only if $1 - p_1 \in E$.*
(ii) *If $p_j \geq \vartheta_d$ for all $j \geq 2$, then E is connected.*
(iii) *For every $0 < \delta < \vartheta_d$, there exists $k = k(\delta)$ such that if $p_{m+r} < \vartheta_d - \delta$, for some $m \geq 2$ and all $0 \leq r \leq k$, then E is not connected.*
(iv) *If $p_j < 1/2$, for some j, then E has at least d^j connected components.*

(v) *If $p_j < 1/2$ for infinitely many j's, then E is a Cantor set.*
(vi) *If p_j is randomly chosen in a way that $(p_i)_{i=1}^{\infty}$ is a sequence of iid random variables with $P(p_i < \vartheta_d) > 0$, then $P(E_{\bar{p}}$ is a Cantor set$) = 1$.*

For the proof of the previous Proposition and the next examples, we stop to mention K and we speak directly of E. This avoids some unnecessary assertions.

Examples : We show here with examples in the case where d is odd that few can be said about the connectedness of E when we have indexes j for which $1/2 \le p_j < \vartheta_d$. Recall that in the case $d = 3$, we have $\theta_3 = \frac{1}{2}$ and $\vartheta_3 = \frac{3}{4}$.

1. Take $p_2 = 2/3$ and $p_j = 3/4$ otherwise. In this case, $\tilde{g}_2(0) = -1/2 = -\theta_3$ thus $\tilde{g}_j(0) = -1/2$ for every j. Since the sequence $(\tilde{g}_j(0))_{j\ge 2}$ is bounded, E is connected.
2. Take $p_2 = 2/3$, $p_3 = 9/14$ and $p_j = 3/4$ otherwise. In this case, $\tilde{g}_2(0) = -1/2$ and $\tilde{g}_3(0) = -\frac{3}{4} < -1/2$, thus $\lim_{j\to\infty} \tilde{g}_j(0) = -\infty$ and E is not connected. Moreover, any other critical point of $\tilde{g}_j$ distinct from 0 is in E by the previous case. Then E has d connected components by the Riemann-Hurwitz Lemma (see Figure 1 above).
3. Take $p_2 = 2/3$, $p_3 = 9/14$, $p_4 = \frac{126}{128}$ and $p_j = 3/4$ otherwise. In this case, $\tilde{g}_2(0) = 1/2$, $\tilde{g}_3(0) = -\frac{3}{4} < -1/2$ and $\tilde{g}_4(0) = -\frac{6}{14} - \frac{1}{63} > -1/2$. Thus $-1/2 < \tilde{g}_j(0) < 0$ for every $j \ge 4$ which implies that $(\tilde{g}_j(0))_{j\ge 2}$ is bounded and E is connected.

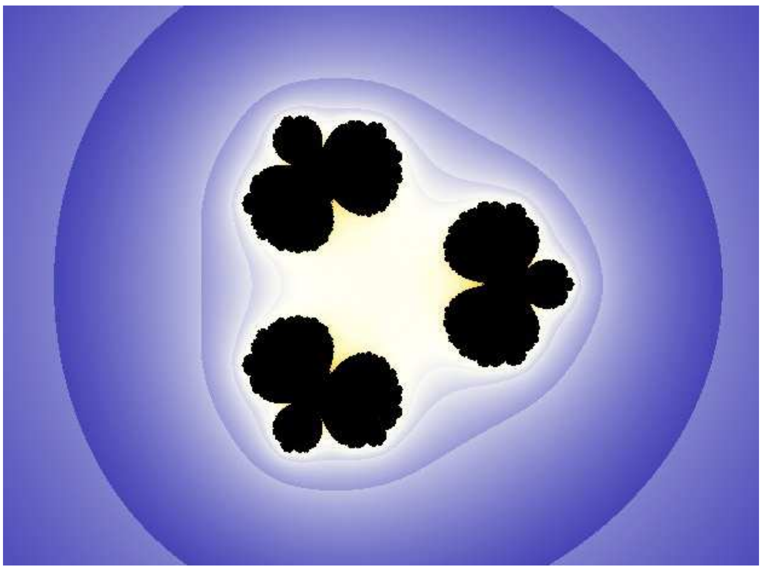

FIGURE 1. In degree 3, E may have a finite number of connected components

Proof. of Proposition 4.6.

Proof of (i): By Riemann-Hurwitz Formula, E is connected, if and only if, E contains all the critical points of $\tilde{f}_j$ for every $j \geq 1$. Since $1-p_1$ is clearly a critical point of every $\tilde{f}_j$, if E is connected then $1-p_1 \in E$. Now suppose that $1-p_1 \in E$ and let z be a critical point of $\tilde{f}_l$, for some fixed $l \geq 2$. By the chain rule for derivatives, there exists $1 \leq m < l$ such that $\tilde{f}_m(z) = 1-p_{m+1}$. Thus, for every $j \geq l$,

$$0 \geq \tilde{f}_j(z) = (f_j \circ ... \circ f_{m+2})(0) \geq (f_j \circ ... \circ f_{m+2})(\tilde{f}_{m+1}(1-p_1)) = \tilde{f}_j(1-p_1)\,,$$

which follows from the fact that $(f_j \circ ... \circ f_m)$ is increasing and $\tilde{f}_{m+1}(1-p_1) = (f_{m+1} \circ ... \circ f_2)(0) < 0$. Since $(\tilde{f}_j(1-p_1))_{j\geq 1}$ is bounded, $(\tilde{f}_j(z))_{j\geq 1}$ is also bounded and $z \in E$. Therefore E is connected.

Proof of (ii): Since d is odd, $1-p_1 \in E$ (equivalent to $0 \in K$), if and only if, $\lim_{j\to\infty} g_j(0) = -\infty$. Let

$$g(x,p) := \frac{x^d - (1-p)}{p},$$

then $\partial_p g(x,p) \geq 0$, for $x < 0$. Thus

$$\tilde{g}_j(0) \geq g^j(0, \vartheta_d)\,,$$

for every j. By Proposition 4.4, we have $-1 < \lim_{n\to\infty} g^j(0,\vartheta_d) < 0$, thus (ii) holds.

Proof of (iii): By Proposition 4.4, we have $\lim_{j\to\infty} g^j(0, \vartheta_d - \delta) = -\infty$. Therefore there exists $k = k(\delta) > 1$ such that

$$g^{k-1}(0, \vartheta_d - \delta) < -1\,.$$

Put

$$f(x,p) := \left(\frac{x-(1-p)}{p}\right)^d.$$

By (4.2), $h_{\vartheta_d-\delta} \circ f^{k-1}(1-p_1, \vartheta_d - \delta) = g^{k-1}(0, \vartheta_d - \delta)$. Hence $f^k(1-p_1, \vartheta_d - \delta) = \left(h_{\vartheta_d-\delta} \circ f^{k-1}(1-p_1, \vartheta_d-\delta)\right)^d < -1$.

Now suppose that $p_{m+r} < \vartheta_d - \delta$, for some $m \geq 2$ and all $0 \leq r \leq k$. Since, f is increasing in both x and p, for $x \leq 0$, then

$$\begin{aligned} \tilde{f}_{m+r}(1-p_1) &= (f_{m+r} \circ ... \circ f_{m+1})(\tilde{f}_m(1-p_1)) \\ &\leq (f_{m+r} \circ ... \circ f_{m+1})(0) \leq f^k(0, \vartheta_d - \delta) \leq -1\,, \end{aligned}$$

for every $j \geq m+r$. Thus $1-p_1 \notin E$ and hence E is not connected.

Proof of (iv): If $p_j < 1/2$ then $f_j(0) < -1$. For every critical point $z \in \mathbb{C}$ of $\tilde{g}_j$, we have that $\tilde{f}_m(z) = 1-p_{m+1}$ for some $0 \leq m < j$, then $\tilde{f}_{j+1}(z) < -1$. Thus $z \notin E$. By Riemann-Hurwitz Formula, E has at least d^j connected components.

Proof of (v): Take $j \to \infty$ in (iv).

Proof of (vi): By continuity of measures, there exists δ such that $P(\rho_i < \vartheta_d - \delta) > 0$. Take $k = k(\delta)$ as in (iii). By Borel Cantelli Lemma, we have that almost surely there exists a sequence $(j_n)_{n\geq 1}$ such that $\lim_{n\to\infty} j_n = \infty$ and $p_{j_n+r} < \vartheta_d - \delta$ for all $0 \leq r \leq k$ and $n \geq 1$. Since, from (iii), we have

$$(f_{j_n+r} \circ ... \circ f_{j_n})(0) < -1\,,$$

following the arguments in (iv), we get that $E_{\bar{\rho}}$ has at least d^{j_n+r} connected components. Since $\lim_{n\to\infty} j_n = \infty$, we have that $E_{\bar{\rho}}$ is Cantor. □

5. Further topological and analytical properties of the spectra

Fix $d \geq 2$. We are now able to identified the sets E and K with fibered Julia sets associated to a suitable fibered polynomial (see [10]). Indeed, let us denote $X = [0,1]^{\mathbb{N}}$, this is a compact set endowed with the product topology.

We define the shift map $\tau : X \to X$ in the standard way : for all $p = (p_1, \ldots, p_n, \ldots) \in X$, $\tau(p) = (p_2, \ldots, p_{n+1} \ldots)$. Let us define the "parameters" of our fibered polynomial map for all $p = (p_n)_{n\geq 1} \in X$

$$a(p) = \frac{1}{p_1} \text{ and } b(p) = -\frac{1-p_1}{p_1}.$$

We introduce the fibered polynomial map of degree d over (X, τ) defined by

$$\begin{array}{rcll} P \;:\; X \times \mathbb{C} & \longrightarrow & X \times \mathbb{C} \\ (p, z) & \longmapsto & \big(\tau(p), P_p(z) = a(p)z^d + b(p)\big)\,, \end{array}$$

Note that P is a skew-product map (*i.e.*the first variable does not depend on the second one!). If P^n is the n-iterate of P then $P^n(p,z) = (\tau^n(p), P_p^n(z))$ where

$$P_p^n(z) = P_{\tau^{n-1}(p)} \circ \ldots \circ P_{\tau(p)} \circ P_p(z), \tag{5.1}$$

Observe that $P_{\tau(p)}^{n+1}$ is exactly the map $\tilde{g}_n$ defined in the previous section.

Now, we need to make the assumption that there exists $\varepsilon > 0$ such that for all $n \geq 2$, $\varepsilon \leq p_n \leq 1$. Thus we restrict τ to the compact set $\widetilde{X} = [\varepsilon, 1]^{\mathbb{N}} \subset X$. With this restriction, a and b turn out to be continuous functions on $\widetilde{X}$. We could also consider the τ-invariant compact set $Y = \overline{\{\tau^n(\omega),\ n \in \mathbb{N}\}}$ for a given $\omega = (p_1, \ldots, p_n, \ldots)$ and consider the restriction $\tau : Y \to Y$.

Whenever $f : X \to \mathbb{C}$ is a continuous map we consider the uniform norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

The first point is to conjugate P to a monic fibered polynomial by a fibered homeomorphism of $X \times \mathbb{C}$ of the form : $(p,z) \mapsto (p, \lambda(p)z)$. This is the objet of the following Proposition

Proposition 5.1. *Assume that* $p_i \in [\varepsilon, 1]$, $\varepsilon > 0$*, then there exist a continuous function* $\lambda : X \to \mathbb{R}^+$ *such that* P *is conjugated by* $(p,z) \mapsto (p, \lambda(p)z)$ *to* $\hat{P}(p,z) = (\tau(p), z^d + c(p))$ *where* $c(p) = \lambda(\tau(p))b(p)$.

Proof. Let us define $\Lambda(p,z) = (p, \lambda(p)z)$. The conjugacy equation reads as follow $\Lambda \circ P(p,z) = \widehat{P} \circ \Lambda(p,z)$. Thus, by identifying the second term of this equation, we obtain for all $(p,z) \in X \times \mathbb{C}$:

$$\lambda(\tau(p))(a(p)z^d + b(p)) = \lambda^d(p)z^d + c(p)$$

This provides

$$\lambda(\tau(p))a(p) = \lambda(p)^d\,, \tag{5.2}$$
$$\lambda(\tau(p))b(p) = c(p)\,. \tag{5.3}$$

To solve the first equation, we simply define $\lambda(p) = \prod_{i=0}^{+\infty} a(\tau^i(p))^{1/d^{i+1}}$. This infinite product is convergent since $1 \leq a(p) = \frac{1}{p_1} \leq \frac{1}{\varepsilon}$ and then

$$0 \leq \sum_{i=0}^{+\infty} \frac{1}{d^{i+1}} \log(a(\tau^i(p)) \leq \frac{1}{d-1} \log(\frac{1}{\varepsilon}) < +\infty\,.$$

Clearly, λ defined by this way, satisfies equation (5.2) and $p \mapsto \lambda(p)$ is continuous on X with $1 \leq \lambda(p) \leq (\frac{1}{\varepsilon})^{\frac{1}{d-1}}$. □

Thus after conjugacy we can assume that the fibered polynomial takes the form $\widehat{P}(p,z) = (\tau(p), z^d + c(p))$. Given a sequence of numbers $(p_i) \in [\varepsilon, 1]$, λ is given by the formula of the previous proposition :

$$\lambda(p) = \prod_{i=1}^{+\infty} \left(\frac{1}{p_i}\right)^{1/d^i}.$$

and $c(p) = b(p)\lambda(\tau(p))$

$$c(p) = -\frac{1-p_1}{p_1} \prod_{i=1}^{+\infty} \left(\frac{1}{p_{i+1}}\right)^{1/d^i}.$$

To avoid heavy notation we still denote $P(p,z)$ the fibered polynomial $\widehat{P} = (\tau(p), z^d + c(p))$ and $P^n(p,z) = (\tau^n(p), P_p^n(z))$. Now we can follow [10] and verify that most of the results there are still valid in our context.

Recall that the global filled-in Julia sets are defined by

$$\mathbf{K} = \Big\{(p,z) \in X \times \mathbb{C} \text{ such that } \sup_{n \in \mathbb{N}} |P_p^n(z)| < +\infty\Big\},$$

and for all $p \in X$

$$K_p = \big\{z \in \mathbb{C} \text{ such that } (p,z) \in K\big\}\,.$$

One has $P(\mathbf{K}) = \mathbf{K}$ and $P_p(K_p) = K_{\tau(p)}$, $\mathbf{K}$ is a compact subset of $X \times \mathbb{C}$, K_p is compact subset of $\mathbb{C}$ bounded by $1 + \max_{p \in X} |c(p)|$. The fibered Julia sets are the topological boundary $J_p = \partial K_p$, and we let $\mathbf{J} = \overline{\bigcup_{p \in X} \{p\} \times J_p}$.

We also consider the Green function of P defined by

$$G(p,z) = G_p(z) := \lim_{n\to+\infty} \frac{1}{d^n} \log_+ |P_p^n(z)|\,,$$

where $\log_+(p) := \max\{0, \log(p)\}$. Then one has Proposition 2.4 in section 2.3 of [10]:

Proposition 5.2. *The map $G : X \times \mathbb{C} \to \mathbb{R}^+$ is continuous and satisfies:*

1. *G_p is harmonic in $\mathbb{C} \setminus K_p$ and K_p is exactly the set of $z \in \mathbb{C}$ such that $G_p(z) = 0$;*
2. *G satisfies the functional equation : $G(P(p,z)) = dG(p,z)$;*
3. *there exist constants A and B such that for all $|z| \geq B$,*

$$\sup_{p\in X} \Big| G_p(z) - \log|z| \Big| \leq \frac{A}{|z|^2}\,. \tag{5.4}$$

In section 2.4 of [10], the continuity of the K_p and J_p with respect to p is considered, see also [5]. Let us denote Comp($\mathbb{C}$) the set of non-empty compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance. Then, by Proposition 2.9 in [10], we have the following result.

Proposition 5.3. *1) The map $p \in X \mapsto K_p \in Comp(\mathbb{C})$ is upper semi-continuous.*
2) The map $p \in X \mapsto J_p \in Comp(\mathbb{C})$ is lower semi-continuous.

Theorem 5.4. *Let $P(p,z) = (\tau(p), z^d + c(p))$ a fibered polynomial over (X,τ). If the continuous map $c : X \mapsto \mathbb{C}$ satisfies,* $\max|c| < (\frac{1}{2})^{\frac{d}{d-1}}$ *then there exists $k > 1$ such that the Julia sets J_p are κ-quasicircles, i.e. J_p is the image of the unit circle by a k-quasiconformal map.*

Concerning quasiconformal mapping and quasiconformal circles we refer to Ahlfors ([1]) where a lot of characterization and properties are provided.

Here we need to adapt the results of [10] where only the degree $d = 2$ was considered.

Proof. The first step is to find an attracting domain. Let $r := (\frac{1}{2})^{\frac{1}{d-1}}$, we claim that the image $P_p(\mathbb{D}(0,r))$ is compactly contained in $\mathbb{D}(0,r)$. Indeed, for all $|z| \leq r$

$$|P_p(z)| \leq |z|^d + |c(p)| < r^d + \left(\frac{1}{2}\right)^{\frac{d}{d-1}} = \left(\frac{1}{2}\right)^{\frac{d}{d-1}} + \left(\frac{1}{2}\right)^{\frac{d}{d-1}} = \left(\frac{1}{2}\right)^{\frac{1}{d-1}}.$$

Next, we will check that this property is sufficient to guarantee that the Julia set is a "uniform" quasi-circle. We follow the main lines in [10] see also [3].

Let us denote $V_p = P_p^{-1}(\mathbb{D}(0,r))$, and $A(p) = V_p \setminus \overline{\mathbb{D}(0,r)}$. Note that V_p is a simply connected domain and thus $A(p)$ is and annulus with bounded modulus from below.

According to lemma (5.5) in [10], there exists $K > 0$ and for all $p \in X$ a K-quasiconformal diffeormorphism $\eta_p : \mathbb{C} \to \mathbb{C}$ such that $\eta_p = Id$ on the complement of V_p, η_p is holomorphic on $\mathbb{D}(0,r)$ and satisfies $\eta_p(c(p)) = 0$.

Let us define $\widetilde{P}_p = \eta_p \circ P_p$ and

$$\widetilde{P}_p^n = \widetilde{P}_{\tau^{n-1}(p)} \circ \ldots \circ \widetilde{P}_p$$

We also consider an ellipse field σ_p by making it circle on $\mathbb{D}(0,r) \cup (\mathbb{C} \setminus K_p)$ and to be invariant under $\widetilde{P}_p$ on

$$\bigcup_{n=0}^{\infty} (\widetilde{P}_p^n)^{-1}(A(\tau^n(p)).$$

The crucial point is that the sets $(\widetilde{P}_p^n)^{-1}(A(\tau^n(p))$ for $n \in \mathbb{N}$ are disjoint. Thus σ_p is well defined and there is only one distortion in the first iteration since $\widetilde{P}_p$ is analytic everywhere except on $A(p)$. Let us denote μ_p the Beltrami coefficient of σ_p, and define $\mu_p = 0$ on any remaining part of $\mathbb{C}$. Hence

$$\|\mu_p\|_\infty \leq \kappa < 1.$$

Let ψ_p be the solution of the associated Beltrami equation $\overline{\psi_p} = \mu_p \psi_p$ with the normalization $\psi_p(0) = 0$ and ψ_p is tangent to the identity at infinity. It is a k-quasiconformal map and by construction, $\psi_{\tau(p)} \circ \widetilde{P}_p \circ \psi_p^{-1}$ turns out to be an analytic map of degree d, with the critical point of order $d-1$ at 0 and also critical value 0. Thus, $\psi_{\tau(p)} \circ \widetilde{P}_p \circ \psi_p^{-1}(z) = z^d$ for all $z \in \mathbb{C}$. But recall, $\widetilde{P}_p = P_p$ on the complement of V_p, this yields :

$$P_p(\psi_p^{-1}(z)) = \psi_{\tau(p)}^{-1}(z^d) \quad \forall |z| \geq r_0$$

and J_p is the image of the unit circle by the k-quasiconformal map ψ_p^{-1}. $\square$

Corollary 5.5. *There exist $0 < \rho < 1$ such that whenever $p_i \in [\rho, 1]$ for all $i \geq 2$, then there exist $\kappa > 1$ such that J_p is κ-quasicircle. Actually, we can take $\rho = 2(\sqrt{2} - 1) \approx 0.828$.*

Proof. It suffices to prove that whenever $p_i \in [\rho, 1]$ with ρ suitable chosen

$$|c(p)| = \frac{1 - p_1}{p_1} \prod_{i=1}^{+\infty} \left(\frac{1}{p_{i+1}} \right)^{1/d^i} < \left(\frac{1}{2} \right)^{\frac{d}{d-1}}. \tag{5.5}$$

Assume $p_i \in [\rho, 1]$ then $1 \leq \frac{1}{p_i} \leq \frac{1}{\rho}$ and

$$|c(p)| \leq \left(\frac{1}{\rho} - 1 \right) \left(\frac{1}{\rho} \right)^{\frac{1}{d-1}} \leq \left(\frac{1}{\rho} - 1 \right) \left(\frac{1}{\rho} \right).$$

Since $\frac{1}{4} < \left(\frac{1}{2}\right)^{\frac{d}{d-1}}$, a sufficient condition that implies inequality (5.5) is

$$\left(\frac{1}{\rho} - 1 \right) \left(\frac{1}{\rho} \right) \leq \frac{1}{4}.$$

In particular when $\rho = 2(\sqrt{2} - 1)$ there is equality, thus (5.5) hold.

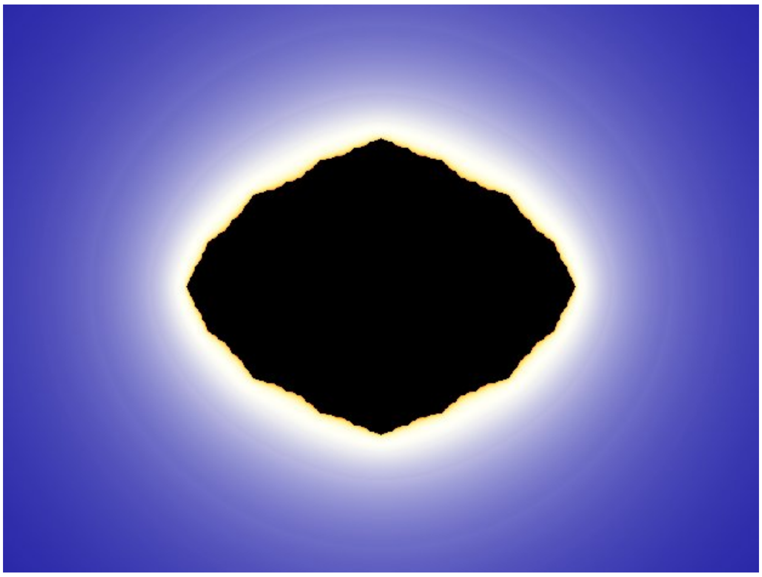

FIGURE 2. A filled-in Julia set which is a quasi-disque, here (p_i) is a sequence of uniform random variables in $[0.83, 9]$

Then the previous Theorem provides the conclusion. □

Remark 5.1. *To be more precise, associated to each d there is a unique solution of the equation*

$$\left(\frac{1}{\rho} - 1\right)\left(\frac{1}{\rho}\right)^{\frac{1}{d-1}} = \left(\frac{1}{2}\right)^{\frac{d}{d-1}}$$

which provides a value $0 < \rho(d) < 1$ a little more accurate.

Finally, we can easily deduce the same result for the spectrum of our Markov chain S_d. Indeed, in Theorem 3.4 we have shown that the spectrum is exactly E_p, the set of points z with bounded orbit that is such that the family $(\tilde{f}_n(z))_{n\in\mathbb{N}}$ is bounded. Then we conjugated $\tilde{f}_n$ to $\tilde{g}_n$ through affine maps : $\tilde{g}_n \circ h_1 = h_{n+1} \circ \tilde{f}_n$. Finally, $\tilde{g}_n$ is conjugated to P_p^n thanks to Proposition 5.1. Thus we have that the following diagram is commutative :

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\tilde{f}_n} & \mathbb{C} \\
{\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_{n+1}} \\
\mathbb{C} & \xrightarrow{\tilde{g}_n} & \mathbb{C} \\
{\scriptstyle z\mapsto\lambda(p)z}\big\downarrow & & \big\downarrow{\scriptstyle z\mapsto\lambda(\tau^n(p))z} \\
\mathbb{C} & \xrightarrow{P_p^n} & \mathbb{C}
\end{array}$$

Since we need to restrict our parameters $p \in [\varepsilon, 1]^{\mathbb{N}}$ clearly $(\tilde{f}_n(z))_{n\in\mathbb{N}}$ is bounded if and only $(P_p^n(\lambda(p)h_1(z)))_{n\in\mathbb{N}}$ is bounded. Let ψ denote the inverse of $z \mapsto \lambda(p)h_1(z)$ which is also an affine map, it follows that E_p is the image of K_p under ψ. Thus the conclusions of Proposition 5.3, Theorem 5.4, and Corollary 5.5 also applied to E_p.

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