HYDRODYNAMIC LIMIT FOR A CLASS OF EXCLUSION TYPE PROCESSES WITH CONDUCTANCES IN DIMENSION GREATER THAN ONE

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Abstract. For a fixed smooth regular hypersurface Λ on the d-dimensional torus, we consider a random walk on the discrete torus of size N with the jump rate to cross the bond connecting x to $x+e_j$ having order $1/N$, if the bond intersect $N\Lambda$, and 1 otherwise. The hypersurface Λ models the effect of a membrane that slows down the passage of particles. For exlusion processes, where particles evolve as random walks associated to Λ, we obtain the hydrodynamic limit whose equation is a d-dimensional version of a parabolic partial diferential equation associated to a Krein-Feller operator.

1. Introduction

Let Π^d be the d-dimensional torus, i.e, $[0,1)^d$ with periodic boundary conditions. Denote by $(e_j)_{j=1}^d$ the canonical base of $\mathbb{R}^d$. Let Λ be a simple closed two times continuously differentiable hypersurface on Π^d and let R_1 and R_2 be the two disjoint open connected components of $\Pi^d-\Lambda$. Denote by $\Pi_N^d=(\mathbb{Z}/N\mathbb{Z})^d$ the discrete torus with N points and put $\Omega_N^d=\{0,1\}^{\Pi_N^d}$. For $u\in\Pi_N^d$ and $1\leq j\leq d$, we use the notation $u_{-j}:=(u_1,...,u_{j-1},u_{j+1},...,u_d)$. During this paper we are going to use (u_j,u_{-j}) to represent $u\in\Pi_N^d$, when we need to put in evidence the kth coordinate of u.

We define the bonds crossing rates as ϑ

$$\xi_{x,j}^N=\begin{cases}\frac{1}{N} & \text{, if } (x/N,(x+e_j)/N)\cap\Lambda\neq\emptyset,\\ & \text{ or } \{x/N,(x+e_j)/N\}\cap\Lambda\neq\emptyset \text{ with } (x/N,(x+e_j)/N)\cap R_2\neq\emptyset,\\ 1 & \text{, otherwise}.\end{cases}$$

for every $j=1,...,d$ and $x\in\Pi_N^d$.

Consider the random walk on $N^{-1}\Pi_N^d$ with generator

$$(\mathbb{L}_N v)(x/N)=\sum_{j=1}^d\Big(\xi_{x,j}^N\Big[v\Big(\frac{x+e_j}{N}\Big)-v\Big(\frac{x}{N}\Big)\Big]+\xi_{x-e_j,j}^N\Big[v\Big(\frac{x-e_j}{N}\Big)-v\Big(\frac{x}{N}\Big)\Big]\Big),$$

for every $v:N^{-1}\Pi_N^d\to\mathbb{R}$ and $x\in\Pi_N^d$. This random walk will be called the **random walk with conductances given by** Λ. Since the transition probability function is symmetric, we have that the uniform distribution on Π_N^d is reversible for the random walk with conductances given by Λ.

The surface Λ represents a permeable membrane which tends to slow down and reflect particles on its neighborhood, creating space discontinuities in the solutions.

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The exclusion process with conductances given by Λ on Ω_N^d is a markov chain with configuration space Ω_N^d and generator given by

$$L_N f(\eta) = \sum_{x\in\Pi_N^d}\sum_{j=1}^{d} \xi_{x,j}^N c_{x,x+j}(\eta)[f(\eta^{x,x+e_j}) - f(\eta)]$$

for every $f : \Omega_N^d \to \mathbb{R}$ and $\eta = (\eta(x))_{x\in\Pi_N^d}$, where

$$c_{x,x+j}(\eta) = \eta(x)[1-\eta(x+e_j)] + \eta(x+e_j)[1-\eta(x)]$$

and

$$\eta^{x,x+e_j}(y) = \begin{cases} \eta(x+e_j) & , \; y = x \\ \eta(x) & , \; y = x+e_j \\ \eta(y) & , \; \text{otherwise}. \end{cases}$$

Here $(\eta_t^N)_{t\geq 0}$ will be used to denote an exclusion process with conductances given by Λ which is a Markov process on Ω_N^d with generator L_N. The Bernoulli product measures on Π_N^d given by

$$\nu_\alpha(\eta) = \prod_{x\in\Pi_N^d} \alpha^{\eta(x)}(1-\alpha)^{1-\eta(x)} \quad \eta \in \Omega_N^d,$$

are reversible for the exclusion process with conductances.

Before we are able to state the hydrodynamic limit, let us discuss how we arrive at the hydrodynamic equation. For each fixed $u \in \Pi^d$ define the strictly increasing functions

$$W_j(v|u_{-j}) = v + F_j(v|u_{-j}) \quad u \in [0,1) \quad j = 1, ..., d\,,$$

where

$$F_j(v|u_{-j}) = \sum_{w\in C_j(u_{-j})} \mathbb{I}_{[w,1)}(v)$$

with $C_j(u_{-j}) = \{w \in [0,1) : (w, u_{-j}) \in \Lambda\}$ Note that

$$\xi_{x,j}^N = \frac{1}{N\Big[W_j\Big(\frac{x_1+1}{N}\big|x_2\Big) - W_j\Big(\frac{x_1}{N}\big|x_2\Big)\Big]}$$

This leads us to think in a hydrodynamic equation of the form $\partial_t\rho = \sum_{j=1}^d \partial_{u_j}\partial_{W_j}$.

Now we consider the operator $\mathcal{U}_\Lambda = \sum_{j=1}^d \partial_{u_j}\partial_{W_j}$. The domain $\mathcal{D}_\Lambda$ of $\mathcal{U}_\Lambda$ is defined as the set of functions $g \in L^2(\Pi^d)$ for which there exist functions $h_j \in L^2(\Pi^d)$, $a_j \in L^2(\Pi^{d-1})$ and $b_j \in L^2(\Pi^{d-1})$, $j = 1, ..., d$, such that

$$\int_0^1 h_j(w, u_{-j})dw = 0\,, \quad \forall\, u \in \Pi^d\,, \quad \text{and } j = 1, ..., d\,,$$

$$\int_{(0,1]} W_j(dy|u_{-k}) \left(\int_0^y h_j(w, u_{-k})dw + b_j(u_{-k})\right) = 0, \quad \forall\, u \in \Pi^d\,,$$

(these are required boundary conditions) and for every $u \in \Pi^d$

$$g(u) = a_j(u_{-j}) + \int_{(0,u_j)\cup A_j(u)} W_j(dy|u_{-j}) \left(\int_0^y h_j(w, u_{-j})dw + b_j(u_{-j})\right), \quad (1.1)$$

where,

$$A_j(u) = \begin{cases} \{u_j\} & , \text{ se } u \in \Lambda \text{ e } \exists\, \epsilon > 0 \text{ tal que } (w, u_{-j}) \in R_2 \;\forall w \in (u_j - \epsilon, u_j) \\ \emptyset & , \text{ se } u \in \Lambda \text{ e } \exists\, \epsilon > 0 \text{ tal que } (w, u_{-j}) \in R_1 \;\forall w \in (u_j - \epsilon, u_j). \end{cases}$$

Therefore, for $g \in \mathcal{D}_\Lambda$ we have $\partial_{u_j}\partial_{W_j} g = h_j$ and we define $\mathcal{U}_\Lambda g = \sum_{j=1}^d h_j$.

Note that every function in $C^2(\Pi^d)$ that is identically zero on Λ with gradient also identically zero on Λ is in $\mathcal{D}_\Lambda$. This would not be an appropriated domain for $\mathcal{U}_\Lambda$, but $\mathcal{D}_\Lambda$ is larger and through solutions of eliptic PDE's we can obtain functions in $\mathcal{D}_\Lambda$ which are discontinuous all along the curve Λ. Indeed we will see soon that $\mathcal{U}_\Lambda$ is the generator of a Markov Process and if $\mathcal{D}_\Lambda$ does not contain functions that are discontinuous on Λ, then particles would be reflected on Λ.

We can show the following result using similar arguments as those presented in [3]:

Theorem 1.1. *The operator $\mathcal{U}_\Lambda : \mathcal{D}_\Lambda \to L^2(\Pi^d)$ have the following properties:*

(a) *The domain $\mathcal{D}_\Lambda$ is dense in $L^2(\Pi^d)$.*
(b) *The operator $\mathcal{U}_\Lambda$ is symmetric and nonpositive. More precisely, for every g_1 and g_2 in $\mathcal{D}_\Lambda$.*

$$-\int_0^1\int_0^1 (\mathcal{U}_\Lambda g_1) g_2 \, du dv \geq 0$$

(c) $\mathrm{I} - \mathcal{U}_\Lambda : \mathcal{D}_\Lambda \to L^2(\Pi^d)$ *is bijective.*
(d) $\mathcal{U}_\Lambda$ *is dissipative.*

By the Hille-Yoshida theorem, $\mathcal{U}_\Lambda$ is the generator of a strongly continuous contraction semigroup in $L^2(\Pi^d)$.

Theorem 1.2. *Fix a continuous function $\rho_0 : \Pi^d \to [0,1]$ and consider a sequence of probability measures ϑ_N on Ω_N^d associated to ρ_0, i.e., for every $\delta > 0$*

$$\lim_{N\to\infty} \vartheta_N \left\{ \left| \frac{1}{N^d} \sum_{x\in\Pi_N^d} H(x/N)\eta(x) - \int_{\Pi^d} H(u)\rho_0(u)du \right| > \delta \right\} = 0 \,.$$

Then, if $\mathbb{P}^N_{\vartheta_N}$ is the distribution of (η_t^N) with initial distribution ϑ_N, for every $t \geq 0$, $\delta > 0$ and every continuous function $H : \Pi^d \to \mathbb{R}$, we have that

$$\lim_{N\to\infty} \mathbb{P}^N_{\vartheta_N} \left\{ \left| \frac{1}{N^d} \sum_{x\in\Pi_N^d} H(x/N)\eta_{tN^2}(x) - \int_{\Pi^d} H(u)\rho(t,u)du \right| > \delta \right\} = 0 \,,$$

where ρ is the unique solution of equation

$$\begin{aligned} &\partial_t \rho = \mathcal{U}_\Lambda \rho \\ &\rho(0,\cdot) = \rho_0(\cdot) \,. \end{aligned} \tag{1.2}$$

To prove the last theorem, we show a uniqueness result for solutions of equation 1.2 and then we follow the method described in [6] which is based on the Γ-convergence of Dirichlet forms. So let us skecth some steps in the proof:

For a Borel measure μ on Π^d and a μ-integrable Borel measurable function $H : \Pi^d \to \mathbb{R}$, we denote by $\mu(H)$ the integral of H with respect to μ. Following the usual method to prove hydrodynamics we consider the empirical measures

$$\pi_t^N = \frac{1}{N^d} \sum_{x\in\Pi_N^d} \eta^N_{tN^2}(x)\delta_{x/N} \,, \ 0 \leq t \leq T \,,$$

where δ_u is the point mass at $u \in \Pi^d$. We need to show that the measure valued process $(\pi_t^N)_{0\leq t\leq T}$ is tight and that its limit points are concentrated on absolutely continuous trajectories $(\pi_t)_{0\leq t\leq T}$ such that

$$\pi_t(H) - \pi_0(H) - \int_0^t \pi_s(\mathcal{U}_\Lambda H)ds = 0\,,\ 0 \leq t \leq T\,, \tag{1.3}$$

for every function $H : \Pi^d \to \mathbb{R}$ in some appropriated core $\mathcal{R}$ of $\mathcal{U}_\Lambda$. This requires, for every $H \in \mathcal{R}$, tightness of $(\pi_t^N(H))_{t\geq 0}$, the proof that

$$\left(\pi_t^N(H) - \pi_0^N(H) - \int_0^t \pi_s^N(\mathbb{L}^N H)ds\right)_{0\leq t\leq T}$$

converges to zero in probability and a result of convergence of $\int_0^t \pi_s^N(\mathbb{L}^N H)ds$ to $\int_0^t \pi_s^N(\mathcal{U}_\Lambda H)ds$.

So we need to deal with the problem of establishing a suitable convergence of $\mathbb{L}^N$ to $\mathcal{U}_\Lambda$ which is non-trivial due to the way we derive with respect to W_1 and W_2 in the boundary of Λ. So here comes the role of the Γ-convergence we now define:

Denote by m_2 the Lebegue measure on Π^d. For $H \in L^2(\Pi^d)$, define

$$H^N(x) = N^d \int_{E_x} H\, dm_2 \quad x \in \Pi_N^d$$

with $E_{x,y} = [\frac{x_1}{N} - \frac{1}{2N}, \frac{x_1}{N} + \frac{1}{2N}) \times [\frac{x_2}{N} - \frac{1}{2N}, \frac{x_2}{N} + \frac{1}{2N})$, and for $v_1, v_2 : \Pi_N^2 \to \mathbb{R}$

$$\langle v_1, v_2\rangle_N = \frac{1}{N^d}\sum_{x\in\Pi_N^d} v_1(x)v_2(x)\,.$$

Put

$$\mathcal{E}_N H = -\langle H^N, \mathbb{L}_N H^N\rangle \ \text{ for } H \in L^2(\Pi^d) \quad \text{and} \quad \mathcal{E}_\Lambda H = -\int_{\Pi^d} H\,\mathcal{U}_\Lambda H\, dm_2 \ \text{ for } H \in \mathcal{D}_\Lambda\,.$$

Proposition 1.3. *$\mathcal{E}_N$ is Γ-convergent to $\mathcal{E}_\Lambda$, i.e, for every $H \in \mathcal{D}_\Lambda$*

- *$\mathcal{E}_\Lambda H \leq \liminf_{N\to\infty} \mathcal{E}_N G_N$, for any sequence $(G_N)_{N\geq 1}$ converging to H in $L^2(\Pi^d)$.*
- *$\mathcal{E}_\Lambda H \geq \limsup_{N\to\infty} \mathcal{E}_N F_N$, for some sequence $(F_N)_{N\geq 1}$ converging to H in $L^2(\Pi^d)$.*

Γ-convergence implies convergence of the minimizers of the Dirichlet forms and this result can be used to show that

$$\sup_{0\leq t\leq T}\left|\pi_t^N(H) - \frac{1}{N^2}\sum_{x\in\Pi_N^d} H_\lambda^N \eta_{tN^2}^N(x)\right|$$

converges to zero in probability for all $H \in \mathcal{R}$ when $N \to \infty$ and $\lambda \to \infty$, where $H_\lambda^N = (\lambda - \mathbb{L}_N)^{-1} S^N H$.

So we define

$$\pi_t^{N,\lambda}(H) = \frac{1}{N^d}\sum_{x\in\Pi_N^d} H_\lambda^N \eta_{tN^2}^N(x)\,,$$

and we work if $\pi^{N,\lambda}$ in place of π_t^N. Then for $H \in \mathcal{R}$ we can prove tightness for $(\pi_t^{N,\lambda} H)_{0\le t\le T}$ which implies tightness for $(\pi_t^N H)_{0\le t\le T}$ and then show, for every limit point of $(\pi_t^N)_{0\le t\le T}$ and for every $G : \Pi^d \to \mathbb{R}$ continous, that

$$\pi_t(G_\lambda) - \pi_0(G_\lambda) - \int_0^t \pi_s(\mathcal{U}_\Lambda G_\lambda) ds = 0\,,\ 0 \le t \le T\,,$$

where $G_\lambda = (\lambda - \mathcal{U}_\Lambda)^{-1} G$. Thus we have (1.3) and all limit points of $(\pi_t^N)_{0\le t\le T}$ is concentrated on solutions of the hydrodynamic equation (1.2).

2. The operator $\mathcal{U}_\Lambda$ (Proof of Theorem 1.1)

Let C_Λ be the space of functions $g : \Pi^d \to \mathbb{R}$ such that

(i) $g_{|\bar{R}_1}$ is continuous;

(ii) $g_{|R_2}$ is uniformly continuous (then, it has a unique continuous extension to $\bar{R}_2$).

Thus if $g \in C_\Lambda$ then the set of discontinuity points of g is a subset of Λ. The space C_Λ is endowed with the sup norm $\|\cdot\|_\infty$.

For each fixed $u \in \Pi^d$ define the strictly increasing functions

$$\tilde{W}_j(v|u_{-j}) = v + \tilde{F}_j(v|u_{-j}) \quad u \in [0,1) \quad j = 1, ..., d\,,$$

where

$$\tilde{F}_j(v|u_{-j}) = \sum_{w\in C_j^+(u_{-j})} \mathbb{I}_{[w,1)}(v) + \sum_{w\in C_j^-(u_{-j})} \mathbb{I}_{(w,1)}(v)$$

with

$$C_j^+(u_{-j}) = \{w \in [0,1) : (w, u_{-j}) \in \Lambda,\ \forall \epsilon > 0 \text{ small enough } (w - \epsilon, u_{-j}) \in R_2\},$$

$$C_j^-(u_{-j}) = \{w \in [0,1) : (w, u_{-j}) \in \Lambda,\ \forall \epsilon > 0 \text{ small enough } (w - \epsilon, u_{-j}) \in R_1\},$$

Note that $C_j(u_{-j}) = C_j^+(u_{-j}) \cup C_j^-(u_{-j})$.

For every $j = 1, ..., d$, define the generalized partial derivative ∂_{W_j} as follows

$$\partial_{W_j} g(u) = \lim_{\epsilon\to 0} \frac{g(u + \epsilon e_j) - g(u)}{\tilde{W}_j(u_j + \epsilon|u_{-j}) - \tilde{W}_j(u_j|u_{-j})}\,, \quad u \in \Pi^d\,,$$

if the above limit exists and is finite. Denote by D_Λ the set of functions in C_Λ such that $\partial_{W_j} g$, is well defined and differentiable in the j-th coordinate with $\partial_{u_j}\partial_{W_j}$ continuous, for all $j = 1, ..., d$. *(verificar a necessidade de ter $\partial_{u_j}\partial_{W_j}$ em C_Λ)*

Define the operator $\mathrm{U}_\Lambda : \mathbb{D}_\Lambda \to C_\Lambda(\Pi^d)$ by

$$\mathrm{U}_\Lambda g \;=\; \sum_{j=1}^d \partial_{u_j}\partial_{W_j} g \;=\; \sum_{j=1}^d \partial_{u_j}(\partial_{W_j} g)\,.$$

By [1, Lemma 0.9 in Appendix], given $g \in C_\Lambda$ and a continuous function h,

$$\partial_{W_j} g(u) \;=\; h(u)$$

for all u in Π^d if and only if

$$g(w_1, u_{-j}) - g(w_2, u_{-j}) \;=\; \int_{B_j(w_1,w_2)} h(v, u_{-j}) dW_j(v) \tag{2.1}$$

for all $w_1 < w_2$ and $u_{-j} \in \Pi^{d-1}$, where

$$B_j(w_1, w_2) = ([w_1, w_2) \cup A_j(w_2, u_{-j})) - A_j(w_1, u_{-j})\,.$$

Note that

$$\int_{(0,1]} h(v, u_{-j})\, dW_j(v) \;=\; 0\,,$$

because $g(0, u_{-j}) = g(1, u_{-j})$ forall $u_{-j} \in \Pi^{d-1}$.

It follows from this observation and the definition of the operator U_Λ that D_Λ is the set of functions g in C_Λ such that, for every $j = 1, ..., d$,

$$g(u) = a_j(u_{-j}) + \int_{(0,u_j)\cup A_j(u)} W_j(dy|u_{-j}) \left(\int_0^y h_j(w, u_{-j})dw + b_j(u_{-j}) \right), \quad (2.2)$$

for some function h_j in C_Λ and two continuous real functions $a_j : \Pi^{d-1} \to \mathbb{R}$ and $b_j : \Pi^{d-1} \to \mathbb{R}$, $j = 1, ..., d$, such that

$$\int_0^1 h_j(w, u_{-j})dw = 0\,, \quad \forall\, u \in \Pi^d\,, \quad \text{and } j = 1, ..., d\,, \quad (2.3)$$

$$\int_{(0,1]} W_j(dy|u_{-k}) \left(\int_0^y h_j(w, u_{-k})dw + b_j(u_{-k}) \right) = 0, \quad \forall\, u \in \Pi^d\,, \quad (2.4)$$

The requirement (2.3) corresponds to the boundary condition $\partial_{W_j} g(0, u_{-j}) = \partial_{W_j} g(0, u_{-j})$ and (2.4) to the boundary condition $g(0, u_{-j}) = g(1, u_{-j})$. One can check that the functions h_j, a_j and b_j are unique.

Lemma 2.1. *The following statements hold.*

(1) *The set* D_Λ *is dense in* $L^2(\Pi^d)$.

(2) *The operator* $\mathrm{U}_\Lambda : \mathrm{D}_\Lambda \to L^2(\Pi^d)$ *is symmetric and nonpositive. More precisely,* $\langle \mathrm{U}_\Lambda f, g\rangle$ *is equal to*

$$-\sum_{k=1}^{d} \int_{\Pi^{d-1}} du_{-k} \int_0^1 F_k(dz|u_{-k})\, ((\partial_{F_k} f)(u_{-k}, z))\, ((\partial_{F_k} g)(u_{-k}, z))$$

for all f, g *in* D_Λ.

(3) *The operator* U_Λ *satisfies a Poincaré inequality: There exists* $C > 0$ *such that*

$$\|g\|_2^2 \le C\langle -\mathrm{U}_\Lambda g, g\rangle + \Big(\int_{\Pi^d} g(u)du \Big)^2$$

for every g *in* D_Λ.

Proof of Lemma 2.1:

Proof of (a): If we take functions with support does not intersect Λ, it is easy to show the density in L^2 of D_Λ. We not only show this fact, but also point the existence of functions in the domain which are smooth in $\Pi^d \backslash \Lambda$ and discontinuos over Λ. Let $\Lambda_\delta = \{x \in \Pi^d;\ \mathrm{dist}(x, \Lambda) < \delta\}$. Choose Φ a partition unity of Λ_δ such that $\mathrm{supp}(\Phi) \subseteq \Lambda_{2\delta}$. Fix $(a_1, \dots, a_d) \in \mathbb{R}^d$ and define $g : [0,1)^d \to \mathbb{R}$, $g(x_1, \dots, x_d) = a_1x_1 + \cdots + a_dx_d$ a linear function. Therefore, the function $g(x_1, \dots, x_d) \cdot \left(\sum_{\phi_i \in \Phi} \phi_i \right)$ is C^∞ and its gradient along the surface Λ is constant and equal to $(a_1, \dots, a_d)$. Then its easy to verify that, if $a_1 = \cdots = a_d = a$,

$$g(x_1, \dots, x_d) \cdot \Big(\sum_{\phi_i \in \Phi} \phi_i \Big) + \mathbf{1}_{R_1}$$

belongs to D_Λ. If we sum a smooth function which support does not intersect $C_{2\Lambda}$, we obtain another function in the domain, and consequently the density, because 2δ can be chosen arbitraly small.

Proof of (b): By definition $\langle \mathrm{U}_\Lambda g_1, g_2 \rangle$ is equal to

$$\sum_{k=1}^{d} \int_{\Pi^d} du (\partial_{u_k} \partial_{F_k} g_1)(u)\, g_2(u) = \sum_{k=1}^{d} \int_{\Pi^{d-1}} du_{-k} \int_0^1 dz (\partial_{u_k} \partial_{F_k} g_1)(u_{-k}, z)\, g_2(u_{-k}, z)$$

that, by the one-dimensional result in ... of [3], can be written as

$$\sum_{k=1}^{d} \int_{\Pi^{d-1}} du_{-k} \int_0^1 F_k(dz|u_{-k})\, (\partial_{F_k} g_1)(u_{-k}, z)\, (\partial_{F_k} g_2)(u_{-k}, z)\, .$$

In particular,

$$\langle \mathrm{U}_\Lambda g, g \rangle = \sum_{k=1}^{d} \int_{\Pi^{d-1}} du_{-k} \int_0^1 F_k(dz|u_{-k})\, ((\partial_{F_k} g)(u_{-k}, z))^2 \geq 0.$$

Proof of (c): *Verificar se a cont. a esquerda em alguns pontos causa problema.* Write

$$\begin{aligned} \int_{\Pi^d} g(u)^2 du - \Big(\int_{\Pi^d} g(u) du \Big)^2 &= \int_{\Pi^d} \Big\{ \int_{\Pi^d} (g(u) - g(v)) dv \Big\}^2 du \\ &\leq \int_{\Pi^d} \int_{\Pi^d} (g(u) - g(v))^2 dv du\, . \end{aligned} \tag{2.5}$$

If we define by induction $\tilde{u}^0 = u$ and $\tilde{u}^k = (\tilde{u}^{k-1}_{-k}, v_k)$, $k = 1, ..., d$, then

$$\begin{aligned} |g(u) - g(v)| &= \Big| \sum_{k=1}^{d} [g(\tilde{u}^k) - g(\tilde{u}^{k-1})] \Big| \\ &\leq \sum_{k=1}^{d} \Big| \int_{(0,1]} \frac{\partial g}{\partial F_k} (\tilde{u}^k_{-k}, z) F_k(dz|(\tilde{u}^k_{-k}) \Big|\, . \end{aligned}$$

Thus, By Cauchy-Schwarz inequality $|g(u) - g(v)|^2$ is bounded above by

$$2^d \sup\{F_k((0,1]|u_{-k}) : \ u \in \Pi^d,\ 1 \leq k \leq d\} \sum_{k=1}^{d} \int_{(0,1]} \Big(\frac{\partial g}{\partial F_k} (\tilde{u}^k_{-k}, z) \Big)^2 F_k(dz|(\tilde{u}^k_{-k})$$

By (2.5) and the previous inequality we have the Poincaré inequality. □

Denote by $\langle \cdot, \cdot \rangle_\Lambda^{1,2}$ the inner product on D_Λ defined by

$$\begin{aligned} \langle f, g \rangle_\Lambda^{1,2} &= \langle f, g \rangle \ + \ \langle -\mathcal{U}_\Lambda f, g \rangle \\ &= \langle f, g \rangle \ + \sum_{j=1}^{d} \int_{\Pi^{d-1}} \int_{(0,1]} (\partial_{W_j} f)(u)\, (\partial_{W_j} g)(u)\, W_j(du_j)\, du_{-j}\ . \end{aligned}$$

Let $H_\Lambda^{1,2}(\Pi^d)$ be the set of all functions g in $L^2(\Pi^d)$ for which there exists a sequence $\{g_n : n \geq 1\}$ in D_Λ such that g_n converges to g in $L^2(\Pi^d)$ and g_n is Cauchy for the inner product $\langle\cdot,\cdot\rangle_\Lambda^{1,2}$. Such sequence $\{g_n\}$ is called admissible for g. For f, g in $H_\Lambda^{1,2}(\Pi^d)$, define

$$\langle f,g\rangle_\Lambda^{1,2} \;=\; \lim_{n\to\infty}\langle f_n,g_n\rangle_\Lambda^{1,2}\;, \tag{2.6}$$

where $\{f_n\}$, $\{g_n\}$ are admissible sequences for f, g, respectively. By [9, Proposition 5.3.3], this limit exists and does not depend on the admissible sequence chosen. Moreover, $H_\Lambda^{1,2}(\Pi^d)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_\Lambda^{1,2}$ just defined is a real Hilbert space.

Denote by $L_\Lambda^2(\Pi^d)$ the Hilbert space generated by the continuous functions endowed with the inner product $\langle\cdot,\cdot\rangle_\Lambda$ defined by

$$\langle f,g\rangle_\Lambda \;=\; \sum_{j=1}^d\int_{\Pi^{d-1}}\int_{(0,1]} f\,g\,W_j(du_j)\,du_{-j}\;.$$

The norm associated to the scalar product $\langle\cdot,\cdot\rangle_\Lambda$ is denoted by $\|\cdot\|_\Lambda$.

Lemma 2.2. *A function g in $L^2(\Pi^d)$ belongs to $H_\Lambda^{1,2}(\Pi^d)$ if and only if there exist G_1, ... ,G_d in $L_\Lambda^2(\Pi^d)$ and functions $a_j\in L^2(\Pi^{d-1})$ such that*

$$\int_{(0,1]} G_j(v,u_{-j})\,dW_j(v) \;=\; 0 \tag{2.7}$$

and

$$g(u) \;=\; a_j(u_{-j}) \;+\; \int_{(0,u_j)\cup A_j(u)} G_j(v,u_{-j})\,dW_j(v) \tag{2.8}$$

Lebesgue almost surely. We denote the generalized partial W_j-derivative G_j of g by $\partial_{W_j}g$. For f, g in $H_\Lambda^{1,2}(\Pi^d)$,

$$\langle f,g\rangle_\Lambda^{1,2} \;=\; \langle f,g\rangle \;+\; \sum_{j=1}^d\int_{\Pi^{d-1}}\int_{(0,1]}(\partial_{W_j}f)(u)\,(\partial_{W_j}g)(u)\,W_j(du_j)\,du_{-j}\;. \tag{2.9}$$

Proof. Fix g in $H_\Lambda^{1,2}(\Pi^d)$. By definition, there exists a sequence $\{g_n:n\geq 1\}$ in D_Λ which converges to g in $L^2(\Pi^d)$ and which is Cauchy in $H_\Lambda^{1,2}(\Pi^d)$. In particular, for every $j=1,...,d$, $\partial_{W_j}g_n$ is Cauchy in $L_\Lambda^2(\Pi^d)$ and therefore converges to some function G_j in $L_\Lambda^2(\Pi^d)$. By (2.4),

$$\int_{(0,1]}(\partial_{W_j}g_n)(v,u_{-j})\,dW_j(v) \;=\; 0$$

for all $n\geq 1$ so that

$$\int_{(0,1]} G_j(v,u_{-j})\,dW_j(v) \;=\; 0\,,$$

which is (2.7).

In order to prove (2.8), denote by a_j^n the continuous functions that satisfy

$$g_n(u)=a_j^n(u_{-j})+f_j^n(u)$$

for every $u\in\Pi^d$, $j=1,...,d$ and $n\geq 1$, where to simplify notation we are writting

$$f_j^n(u)=\int_{(0,u_j)\cup A_j(u)}(\partial_{W_j}g_n)(v,u_{-j})\,dW_j(v)\,.$$

Now, apply the Cauchy-Schwarz inequality to obtain that

$$\left\| f_j^n - \int_{(0,u_j)\cup A_j(u)} G_j(v,u_{-j})\, dW_j(v) \right\|_2 \leq \|\partial_{W_j} g_n - G_j\|_\Lambda\,.$$

Therefore

$$(f_j^n(u))_{n\geq 1} \text{ converges in } L^2(\Pi^d) \text{ to } \int_{(0,u_j)\cup A_j(u)} G_j(v,u_{-j})\, dW_j(v)\,.$$

for every $j = 1,...d$. Therefore, $(a_j^n(\cdot))_{n\geq 1}$ also converges in $L^2(\Pi^{d-1})$ and we denote its limit by $a_j(\cdot)$. Then (2.8) follows.

We have that (2.9) is a consequence of the established convergence results. □

Lemma 2.3. *The embedding $H^{1,2}_\Lambda(\Pi^d) \subset L^2(\Pi^d)$ is compact.*

Proof. Consider a sequence $\{g_n : n \geq 1\}$ bounded in $H^{1,2}_\Lambda(\Pi^d)$. We need to prove the existence of a subsequence $\{g_{n_k} : k \geq 1\}$ which converges in $L^2(\Pi^d)$.

By the previous lemma, g_n satisfies (2.7) and (2.8) for some $a_j^n \in L^2(\Pi^{d-1})$ and with G_j replaced by $\partial_{W_j} g_n$ which belongs to $L^2_\Lambda(\Pi^d)$. Moreover, $\|\partial_{W_j} g_n\|_\Lambda \leq \|u_n\|^{1,2}_\Lambda$. The sequences $\{\partial_{W_j} g_n\}_{n\geq 1}$, $j = 1,...,d$, are therefore bounded in $L^2_\Lambda(\Pi^d)$. Also, by Schwarz inequality, the sequence $\int_{(0,u_j)\cup A_j(u)} (\partial_{W_j} g_n)(v,u_{-j})\, dW_j(v)$ is bounded in $L^2(\Pi^d)$. Therefore, it is also clear that $(a_j^n)_{n\geq 1}$, $j = 1,...,d$, are also bounded sequences in $L^2(\Pi^{d-1})$.

Since $\{\partial_{W_j} g_n\}$ is a bounded sequence in $L^2_\Lambda(\Pi^d)$ which is separable, there exists a subsequence $\{n_k\}$ such that $\partial_{W_j} g_{n_k}$ converges weakly in $L^2_\Lambda(\Pi^d)$ to a limit denoted by G_j. As in the proof of Lemma 2.2, it follows that

$$\int_{(0,u_j)\cup A_j(u)} (\partial_{W_j} g_{n_k})(v,u_{-j})\, dW_j(v)$$

converges in $L^2(\Pi^d)$ to

$$\int_{(0,u_j)\cup A_j(u)} G_j(v,u_{-j})\, dW_j(v)\,.$$

To make the rest of the proof simpler we suppose that the convergence just stated holds for $n_k = n$.

Therefore, To complete the proof we have to show that there exists a subsequence $\{n_k\}$ such that $(a_j^{n_k})$ converges in $L^2(\Pi^{d-1})$ for every $j = 1,...,d$. To show this, fix $i \neq j$ in $\{1,...,d\}$ and note that

$$\begin{aligned} f_{i,j}^n(u) &= a_i^n(u_{-i}) - a_j^n(u_{-j}) \\ &= \int_{(0,u_i)\cup A_i(u)} (\partial_{W_i} g_n)(v,u_{-i})\, dW_i(v) - \int_{(0,u_j)\cup A_j(u)} (\partial_{W_j} g_n)(v,u_{-j})\, dW_j(v)\,. \end{aligned}$$

Thus $f_{i,j}^n$ converges in $L^2(\Pi^d)$ to some function $f_{i,j}$. In particular, we can fix a subsequence $\{n_k\}$ such that the convergence of $f_{i,j}^{n_k}$ to $f_{i,j}$ holds Lebesgue almost surely. By Fubini's theorem, we have that the convergence holds u_{-i} and u_{-j} almost surely for every pair $i,j \in \{1,...,d\}$. This implies that there exist functions $a_j(u_{-j})$, $j = 1,...,d$, such that $f_{i,j}(u) = a_i(u_{-i}) - a_j(u_{-j})$ for every pair $i,j \in \{1,...,d\}$ and $a_j^{n_k} \to a_j$ almost surely for every $j = 1,...,d$.

It remains to show that we can take a subsequence $\{n_{k'}\}$ of $\{n_k\}$ such that $a_j^{n_{k'}} \to a_j$ in $L^2(\Pi^{d-1})$. From here, we denote by $u_{-i,-j} \in \Pi^{d-2}$ the vector obtained from $u \in \Pi^d$ by removing the i-th and j-th coordinate. Since

$$\int ... \int_{\Pi^d} [(a_i^n(u_{-i}) - a_i(u_{-i})) - (a_j^n(u_{-j}) - a_j^n(u_{-j}))]^2 du_1...du_d \to 0 \ \forall i,j \quad (2.10)$$

and $a_i^n(u_{-i}) - a_i(u_{-i})$ goes to zero almost surely, we have that there exists a subsequence $\{n_k^1\}$ of $\{n_k\}$ such that, $\forall i \neq j$, $u_{-i,-j}$ almost surely

$$\int_0^1 [a_i^{n_k^1}(u_{-i}) - a_i(u_{-i})]^2 du_j \to 0\,. \quad (2.11)$$

We proceed with an analogous argument to show that there exists a subsequence $\{n_k^2\}$ of $\{n_k^1\}$ such that, $\forall i \neq j \neq l$, almost surely with respect to $(u_k\,,\ k \neq i,j,l)$

$$\int\int_{\Pi^2} [a_i^{n_k^2}(u_{-i}) - a_i(u_{-i})]^2 du_j du_l \to 0\,. \quad (2.12)$$

Indeed, from (2.10), there exists a subsequence $\{n_k^2\}$ of $\{n_k^1\}$ such that, $\forall i \neq j \neq l$, almost surely with respect to $(u_k\,,\ k \neq i,j,l)$

$$\int\int_{\Pi^2} [(a_i^{n_k^2}(u_{-i}) - a_i(u_{-i})) - (a_j^{n_k^2}(u_{-j}) - a_j(u_{-j}))]^2 du_j du_l \to 0\,. \quad (2.13)$$

The last integral can be written as the sum of three terms: the first term is the integral in 2.11 replacing i with j and j with l, and thus goes to zero almost surely; the second term is

$$2\int\int_{\Pi^2} [a_i^{n_k^2}(u_{-i}) - a_i(u_{-i})][a_j^{n_k^2}(u_{-j}) - a_j(u_{-j})] du_j du_l \to 0\,,$$

whose absolute value is bounded above by

$$2\left(\int_0^1 [a_j^{n_k^2}(u_{-j}) - a_j(u_{-j})]^2 du_l\right)^{\frac{1}{2}} \left(\int\int_{\Pi^2} [a_i^{n_k^2}(u_{-i}) - a_i(u_{-i})]^2 du_j du_l\right)^{\frac{1}{2}},$$

which, again by 2.11, goes to zero almost surely; and the third term is equal to 2.12 which is the term we are interested, and can be written as the sum of 2.13 with the other terms in its expansion. Therefore 2.12 holds true.

If we keep recursively increasing the number of variables in which we are integrating $[a_i^n(u_{-i}) - a_i(u_{-i})]$ we arrive at a subsequence $\{n_k^{d-1}\}$ such that $a_i^{n_k^{d-1}}$ converges to a_i in $L^2(\Pi^{d-1})$ for every $i = 1, ..., d$. □

Let $\mathcal{D}_\Lambda$ be the set of functions g in $H_\Lambda^{1,2}(\Pi^d)$ for which there exist h_j in $L^2(\Pi^d)$, $j = 1, ..., d$, such that

$$\int_{\Pi^{d-1}} \int_{(0,1]} (\partial_{W_j} f)(u)\,(\partial_{W_j} g)(u)\,W_j(du_j) du_{-j} \;=\; -\langle f, h_j\rangle \quad (2.14)$$

for all f in $H_\Lambda^{1,2}(\Pi^d)$. Since, for $g \in \mathrm{D}_\Lambda$,

$$\int_{\Pi^{d-1}} \int_{(0,1]} (\partial_{W_j} f)(u)\,(\partial_{W_j} g)(u)\,W_j(du_j) du_{-j} \;=\; -\,\langle f\,,\,\partial_{u_j}\partial_{W_j} g\rangle$$

we have that $\mathrm{D}_\Lambda \subset \mathcal{D}_\Lambda \subset H_\Lambda^{1,2}(\Pi^d)$. Moreover, for each $g \in \mathcal{D}_\Lambda$, the functions h_j are uniquely determined because $H_\Lambda^{1,2}(\Pi^d) \supset \mathrm{D}_\Lambda$ is dense in $L^2(\Pi^d)$. We are going to show in the next result that $\mathcal{D}_\Lambda$ is the proper domain for $\mathcal{U}_\Lambda$ in the sense of the definition previous to Theorem 1.1.

Lemma 2.4. *The domain $\mathcal{D}_\Lambda$ consists of all functions g in $L^2(\Pi^d)$ such that*

$$g(u) = a_j(u_{-j}) + \int_{(0,u_j)\cup A_j(u)} W_j(dy)\left(\int_0^y h_j(w,u_{-j})dw + b_j(u_{-j})\right), \quad (2.15)$$

for every $u \in \Pi^d$, where $h_j \in L^2(\Pi^d)$, $a_j \in L^2(\Pi^{d-1})$ and $b_j \in L^2(\Pi^{d-1})$, $j = 1,...,d$, and they satisfy

$$\int_0^1 h_j(w,u_{-j})dw = 0\,, \quad \text{for almost all } u_{-j} \in \Pi^{d-1}\,, \quad \text{and } j = 1,...,d\,, \quad (2.16)$$

$$\int_{(0,1]} W_j(dy)\left(\int_0^y h_j(w,u_{-k})dw + b_j(u_{-k})\right) = 0, \quad \text{for almost all } u_{-j} \in \Pi^{d-1}\,. \quad (2.17)$$

Moreover, in this case,

$$-\int_{\Pi^{d-1}}\int_{(0,1]} (\partial_{W_j} f)(u)\,(\partial_{W_j} g)(u)\,W_j(du_j)du_{-j} \;=\; \langle f, h_i\rangle \quad (2.18)$$

for all f in $H^{1,2}_\Lambda(\Pi^d)$ and $j = 1,...,d$.

Proof. We first show that any function g in $L^2(\Pi^d)$ with the properties listed in the statement of the lemma belongs to $\mathcal{D}_\Lambda$. So take g satisfying (2.15). By the one-dimensional arguments presented in [3], we have, for all f in $H^{1,2}_\Lambda(\Pi^d)$ and $j = 1,...,d$, that

$$-\int_{(0,1]} (\partial_{W_j} f)(v,u_{-j})\,(\partial_{W_j} g)(v,u_{-j})\,W_j(dv) \;=\; \int_{(0,1]} f(v,u_{-j})\,h_j(v,u_{-j})\,dv$$

holds for almost all $u \in \Pi^d$. Therefore, we have (2.18) and, by definition, $g \in \mathcal{D}_\Lambda$.

Conversely, assume that g belongs to $\mathcal{D}_\Lambda$ and take $h_j \in L^2(\Pi^d)$, $j = 1,..,d$, such that

$$-\int_{\Pi^{d-1}}\int_{(0,1]} (\partial_{W_j} f)(u)\,(\partial_{W_j} g)(u)\,W_j(du_j)du_{-j} \;=\; \langle f, h_j\rangle \quad (2.19)$$

for all f in $H^{1,2}_\Lambda(\Pi^d)$. By Lemma 2.2, we have that

$$g(u) \;=\; a_j(u_{-j}) \;+\; \int_{(0,u_j)\cup A_j(u)} (\partial_{W_j} g)(v,u_{-j})\,dW_j(v)\,,$$

for every $j = 1,..,d$. Now we fix j. We have to show that for some $b_j \in L^2(\Pi^{d-1})$, (2.16) and (2.17) hold and

$$(\partial_{W_j} g)(u) = \int_0^{u_j} h_j(w,u_{-j})dw + b_j(u_{-j}) \quad (2.20)$$

for almost every $u \in \Pi^d$. We have that (2.17) follows from the previous identity and (2.7) in Lemma 2.2.

To prove (2.16), let $\{g_n : n \geq 1\}$ be an admissible sequence in D_Λ for g. From (b) in Lemma 2.1, we have that for each $f \in \mathrm{D}_\Lambda(\Pi^d)$,

$$-\int_{(0,1]} (\partial_{W_j} f)(w,u_{-j})\,(\partial_{W_j} g_n)(w,u_{-j})\,W_j(dw) =$$

$$= \int_{(0,1]} f(w,u_{-j})\,(\partial_{u_j}\partial_{W_j} g_n)(w,u_{-j})dw\,, \quad (2.21)$$

for almost all $u \in \Pi^d$ and $\partial_{W_j} g_n$ can be written as

$$\int_0^{u_j} (\partial_{u_j}\partial_{W_j} g_n)(w,u_{-j})dw + b_j^n(u_{-j})$$

Since D_Λ is dense in $H^{1,2}_\Lambda(\Pi^d)$, identity (2.21) holds for every $f \in H^{1,2}_\Lambda(\Pi^d)$. Following the proof of Lemma 2.2, The first integral in (2.21) converges in $L^2(\Pi^{d-1})$ to

$$-\int_{(0,1]} (\partial_{W_j} f)(w,u_{-j})\,(\partial_{W_j} g)(u)\,W_j(du_j)\,.$$

Therefore

$$\int_{\Pi^{d-1}} \left| \int_{(0,1]} f(w,u_{-j})\,(\partial_{u_j}\partial_{W_j} g_n)(w,u_{-j})dw - \int_{(0,1]} f(w,u_{-j})\,h_j(w,u_{-j})dw \right| du_{-j}\,,$$

converges to zero as n goes to ∞. If $f=1$, since

$$\int_{(0,1]} (\partial_{u_j}\partial_{W_j} g_n)(w,u_{-j})dw = 0\,,$$

u_{-j} almost surely, then passing to a subsequence if necessary gives (2.16). If $f(u) = \mathbb{I}_{(0,y)\cup A_j(y,u_{-j})}(u)$, then an analogous argument as above allow us to show that there exists a subsequence $\{n_k\}$ such that

$$\lim_{n\to\infty} \int_0^y (\partial_{u_j}\partial_{W_j} g_n)(w,u_{-j})dw \;=\; \int_0^y h_j(w,u_{-j})dw = 0\,,$$

u_{-j} almost surely. Therefore for almost all $u_{-j} \in \Pi^{d-1}$, b_j^n is Cauchy, and we denote its limit by b_j. Note that $b_j \in L^2(\Pi^{d-1})$ with

$$\|b_j\| \le \|\partial_{W_j} g_n\|_\Lambda \,+\, \|h_j\|\,.$$

This gives (2.20). □

Recall that we denote by I the identity in $L^2(\Pi^d)$. By Lemma 2.1, the symmetric operator $(\mathrm{I}-\mathrm{L}_\Lambda) : \mathrm{D}_\Lambda \to L^2(\Pi^d)$, is strongly monotone:

$$\langle (\mathrm{I}-\mathrm{L}_\Lambda)g, g\rangle \;\ge\; \langle g,g\rangle$$

for all g in D_Λ. Denote by $\mathcal{T}_1 : \mathcal{D}_\Lambda \to L^2(\Pi^d)$ its Friedrichs extension, defined as $\mathcal{T}_1 g = g - \sum_{j=1}^d h_j$, where h_j are the functions in $L^2(\Pi^d)$ given by (2.14). By Theorem 5.5.a in [9], $\mathcal{T}_1$ is self-adjoint, bijective and

$$\langle \mathcal{T}_1 g, g\rangle \;\ge\; \langle g,g\rangle \tag{2.22}$$

for all g in $\mathcal{D}_\Lambda$. Note that the Friedrichs extension of the strongly monotone operator $(\lambda\mathrm{I}-\mathrm{L}_\Lambda)$, $\lambda>0$, is $\mathcal{T}_\lambda = (\lambda-1)\mathrm{I} + \mathcal{T}_1 : \mathcal{D}_\Lambda \to L^2(\Pi^d)$.

Define $\mathcal{U}_\Lambda : \mathcal{D}_\Lambda \to L^2(\Pi^d)$ by $\mathcal{U}_\Lambda = \mathrm{I} - \mathcal{T}_1$. In view of (2.14), $\mathcal{U}_\Lambda g = q$ if and only if $q = \sum_{j=1}^d h_j$ with

$$-\int_{\Pi^{d-1}} \int_{(0,1]} (\partial_{W_j} f)(u)\,(\partial_{W_j} g)(u)\,W_j(du_j)du_{-j} \;=\; \langle f, h_j\rangle$$

for all f in $H^{1,2}_\Lambda(\Pi^d)$ and $j = 1,\dots,d$. In particular by Lemma ?? (b) $\mathcal{U}_\Lambda g = \mathrm{L}_\Lambda g$ for all g in D_Λ. Moreover, if a function g in $\mathcal{D}_\Lambda$ is represented as in Lemma 2.4, $\mathcal{U}_\Lambda g = \sum_{j=1}^d h_j$. This identity together with the identification of the space $\mathcal{D}_\Lambda$ provides the alternative definition of the operator $\mathcal{U}_\Lambda$ presented just before the statement of Theorem 1.1.

Proof of Theorem 1.1. It follows from Lemma 2.1 (a) that the domain $\mathcal{D}_\Lambda$ is dense in $L^2(\Pi^d)$ because $\mathrm{D}_\Lambda \subset \mathcal{D}_\Lambda$. This proves (a).

By definition, $\mathrm{I}-\mathcal{U}_\Lambda = \mathcal{T}_1 : \mathcal{D}_\Lambda \to L^2(\Pi^d)$, which have been shown to be bijective. This proves (b).

The self-adjointness of $\mathcal{U}_\Lambda : \mathcal{D}_\Lambda \to L^2(\Pi^d)$ follows from the one of $\mathcal{T}_1$ and the definition of $\mathcal{U}_\Lambda$ as $\mathrm{I} - \mathcal{T}_1$. Moreover, from (2.22) we obtain that $\langle -\mathcal{T}_\Lambda f, f\rangle \geq 0$ for all f in $\mathcal{D}_\Lambda$.

To prove (d), fix a function g in $\mathcal{D}_\Lambda$, $\lambda > 0$ and let $f = (\lambda \mathrm{I} - \mathcal{U}_\Lambda)g$. Taking the scalar product with respect to g on both sides of this equation, we obtain that

$$\lambda\langle g, g\rangle \;+\; \langle -\mathcal{U}_\Lambda g, g\rangle \;=\; \langle g, f\rangle \;\leq\; \langle g, g\rangle^{1/2}\, \langle f, f\rangle^{1/2} \,.$$

Since g belongs to $\mathcal{D}_\Lambda$, by (c), the second term on the left hand side is positive. Thus, $\|\lambda g\| \leq \|f\| = \|(\lambda \mathrm{I} - \mathcal{U}_\Lambda)g\|$.

We have already seen that the operator $(\mathrm{I}-\mathrm{U}_\Lambda) : \mathrm{D}_\Lambda \to L^2(\Pi^d)$ is symmetric and strongly monotone. By Lemma 2.3, the embedding $H^1_2(\Pi^d) \subset L^2(\Pi^d)$ is compact. Therefore, by [9, Theorem 5.5.c], the Friedrichs extension of $(\mathrm{I} - \mathrm{U}_\Lambda)$, denoted by $\mathcal{T}_1 : \mathcal{D}_\Lambda \to L^2(\Pi^d)$, satisfies claims (e) and (f) with $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_n \uparrow \infty$. In particular, the operator $-\mathcal{U}_\Lambda = \mathcal{T}_1 - \mathrm{I}$ has the same property with $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_n \uparrow \infty$. Since 0 is an eigenvalue of $-\mathcal{U}_\Lambda$ associated at least to the constants, (e) and (f) are in force. □

It follows also from [9, Theorem 5.5.c] that f_n belongs to $H^1_2(\Pi^d)$ for all n.

3. Hydrodynamic Limit

Proposition 3.1. *The sequence of processes* $\{\pi^N_t\,;\, 0 \leq t \leq T\}$ *is tight in the uniform topology, where* π^N_t *is the empirical measure obtained by the exclusion processes with conductances.*

Proof: It is enough to prove that $\{\langle \pi^N_s, H\rangle\,;\, 0 \leq s \leq t\}$ is tight for a dense family of smooth functions $H : \Pi^d \to \mathbb{R}$. By Dynkyn's formula (or apendix in [5]),

$$M^N_t = \langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t \mathbb{L}_N \langle \pi^N_s, H\rangle ds$$

is a martingale. We will show that the quadratic variation of this martingale goes to zero uniformly when N increases, which gives that M^N_t converges in L^2 to zero. By Doob's Inequality, M^N_t is tight.

Put $F = \langle \pi^N_s, H\rangle$. The quadratic variation of M^N_t is given by $\int_0^t (\mathbb{L}_N F^2 - 2F\mathbb{L}_N F)ds$ (see [5] or Revuz-Yor). The term inside the last integral is

$$
\begin{aligned}
\mathbb{L}_N F^2 - 2F\mathbb{L}_N F \;&=\; \sum_i \sum_x N^2 \xi^N_{x,x+e_i}[F^2(\eta_s^{x,x+e_i}) - F^2(\eta_s)] \\
&\quad -2F(\eta_s)\sum_i \sum_x N^2 \xi^N_{x,x+e_i}[F(\eta_s^{x,x+e_i}) - F(\eta_s)] \\
&=\; \sum_i \sum_x N^2 \xi^N_{x,x+e_i}[F(\eta_s^{x,x+e_i}) - F(\eta_s)]^2 \\
&=\; \frac{1}{N^{2d}} \sum_i \sum_x N^2 \xi^N_{x,x+e_i}(\eta_s(x+e_i) - \eta_s(x))^2 \Big(H(\frac{x+e_i}{N}) - H(\frac{x}{N}) \Big)^2 \\
&=\; \frac{1}{N^{2d}} \sum_{i, x\cap\Lambda=\emptyset} N^2 \xi^N_{x,x+e_i}(\eta_s(x+e_i) - \eta_s(x))^2 \Big(H(\frac{x+e_i}{N}) - H(\frac{x}{N}) \Big)^2 \\
&\quad + \frac{1}{N^{2d}} \sum_{i, x\cap\Lambda\neq\emptyset} N^2 \xi^N_{x,x+e_i}(\eta_s(x+e_i) - \eta_s(x))^2 \Big(H(\frac{x+e_i}{N}) - H(\frac{x}{N}) \Big)^2 \\
&=\; \frac{1}{N^{2d}} \sum_{i, x\cap\Lambda=\emptyset} (\eta_s(x+e_i) - \eta_s(x))^2 \Bigg(\frac{H(\frac{x+e_i}{N}) - H(\frac{x}{N})}{N^{-1}} \Bigg)^2 \\
&\quad + \frac{1}{N^{2d-1}} \sum_{i, x\cap\Lambda\neq\emptyset} (\eta_s(x+e_i) - \eta_s(x))^2 \Bigg(\frac{H(\frac{x+e_i}{N}) - H(\frac{x}{N})}{N^{-1}} \Bigg)^2 \\
&\leq\; C \Bigg(\frac{1}{N^{2d}} \sum_{i, x\cap\Lambda=\emptyset} 1 + \frac{1}{N^{2d-1}} \sum_{i, x\cap\Lambda\neq\emptyset} 1 \Bigg)
\end{aligned}
$$

for some constant $C > 0$ depending on $H \in C^1$. The first sum is $O(N^d)$, which divided by N^{2d} goes to zero. Because Λ is smooth, the number of terms in the second sum is $\mathrm{O}(N^{d-1})$, and divided by N^{2d-1} also goes to zero.

Thus, by Doob inequality, for every $\delta > 0$,

$$
\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\leq t\leq T} \big|M_t^N\big| \;>\; \delta \Big] \;=\; 0\,.
$$

In particular, the sequence of martingales $\{M_t^N : N \geq 1\}$ is tight for the uniform topology. One must still examine the integral part in the martingale to conclude the tightness of the process $\langle \pi_t^N, H\rangle$. Using changing of variables and $\xi^N_{x,y} = \xi^N_{y,x}$,

$$
\begin{aligned}
\int_r^t N^2 \mathbb{L}_N F ds &= \frac{1}{N^d}\int_r^t N^2 \sum_{x,i} \xi^N_{x,x+e_i}\Big[\eta_s(x+e_i)H(\frac{x}{N}) + \eta_s(x)H(\frac{x+e_i}{N}) \\
&\quad -\eta_s(x+e_i)H(\frac{x+e_i}{N}) + \eta_s(x)H(\frac{x}{N})\Big]\, ds \\
&= -\sum_x \frac{\xi^N_{x,x+e_i}}{N^d}\int_r^t \eta_s(x)\left[\sum_i \frac{H(\frac{x+e_i}{N}) - H(\frac{x}{N})}{\frac{1}{N^2}}\right] ds \\
&= -\sum_{x;\ \mathrm{dist}(x,\Lambda)>\frac{1}{N}} \frac{1}{N^d}\int_r^t \eta_s(x)\left[\sum_i \frac{H(\frac{x+e_i}{N}) - H(\frac{x}{N})}{\frac{1}{N^2}}\right] ds \\
&\quad -\sum_{x;\ \mathrm{dist}(x,\Lambda)\leq\frac{1}{N}} \frac{\xi^N_{x,x+e_i}}{N^d}\int_r^t \eta_s(x)\left[\sum_i \frac{H(\frac{x+e_i}{N}) - H(\frac{x}{N})}{\frac{1}{N^2}}\right] ds
\end{aligned}
$$

Because $|\xi^N_{x,x+e_i}| \leq 1$, the absolute values of both sums above are bounded by $C(H)(t-s)$ and thus we have the tightness of the integral term, and consequently the tightness of $\langle \pi^N_t, H\rangle$.

3.1. **The hydrodynamic equation.** Consider a bounded density profile $\gamma : \Pi^d \to \mathbb{R}$. A bounded function $\rho : [0,T]\times\Pi^d \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation

$$
\begin{cases} \partial_t \rho \;=\; \mathcal{U}_\Lambda \rho \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \end{cases} \tag{3.1}
$$

if for all functions H in $\mathcal{D}_\Lambda$, all $t>0$,

$$
\langle \rho_t, H\rangle \;-\; \langle \gamma, H\rangle \;=\; \int_0^t \langle \rho_s, \mathcal{U}_\Lambda H\rangle\, ds\ ,
$$

where ρ_t is the notation for $\rho(t,\cdot)$. We prove in Subsection 5.3 uniqueness of weak solutions. Existence follows from the tightness in Section (5.2).

4. Γ-convergence

In this section, we will present all needed tools from Γ-convergence to attain the hydrodynamical limit.

Definition 4.1. *Given $G \in L^2(\Pi^d)$, define the projection $S_N : L^2(\Pi^d) \to L^2(\Pi^d_N)$ by*

$$
S_N G(x) := \frac{1}{|A^N_x|}\int_{A_x} G(y)dy,
$$

where $A^N_x := \begin{cases} \{z\in T^d; |z-y|_{max} < 1/N\}\cap R_1, & \text{if } x\in R_1, \\ \{z\in T^d; |z-y|_{max} < 1/N\}\cap R_2, & \text{if } x\in R_2. \end{cases}$

For two functions $v_1, v_2 : \Pi^d_N \to \mathbb{R}$, we define $\langle v_1, v_2\rangle_N := \frac{1}{N^d}\sum_{x\in\Pi^d_N} v_1(x)v_2(x)$.

Put $\mathcal{E}_N(G) = -\langle S_N G, \mathbb{L}_N S_N G\rangle_N$ for $G \in L^2(\Pi^d)$ and $\mathcal{E}(G) = -\int_{\Pi^d} G\,\mathcal{U}_\Lambda G\, dx$ for $G \in \mathcal{D}_\Lambda$.

Proposition 4.2. *For any $G \in L^2(\Pi^d)$, S_N is close to isometry, or else,*

$$\lim_{N\to\infty} \langle S_N G, S_N G\rangle_N = \|G\|_2^2.$$

Proof. The result is straightfoward for continuous functions. Let $\mathcal{F}_N$ denotes the σ-algebra generated by the partition of Π^d in the sets A_x^N. An easy calculation shows that

$$\langle S_N G, S_N G\rangle_N - \|G\|_2^2 = \|G - \mathbb{E}[G|\mathcal{G}_N]\|_2^2 \ , \tag{4.1}$$

where the expectation is to Lebesgue measure. For a given function $G \in L^2(\Pi^d)$, let be $F : \Pi^d \to \mathbb{R}$ a continuous function such that $\|F - G\|_2 \le \epsilon$. Then,

$$\|F - \mathbb{E}[F|\mathcal{F}_N]\|_2 \quad \le \quad \|F - G\|_2 + \|G - \mathbb{E}[G|\mathcal{F}_N]\|_2 + \|\mathbb{E}[G - F|\mathcal{F}_N]\|_2$$

and applying Jensen inequality to the last term we get the desired convergence. □

Proposition 4.3. *$\mathcal{E}_N$ is Γ-convergent to $\mathcal{E}_\Lambda$, i.e, for every $G \in \mathcal{D}_\Lambda$*

(1) *$\mathcal{E}(G) \le \liminf_{N\to\infty} \mathcal{E}_N(G_N)$, for* ***any*** *sequence $(G_N)_{N\ge1}$ converging to G in $L^2(\Pi^d)$.*
(2) *$\mathcal{E}(G) \ge \limsup_{N\to\infty} \mathcal{E}_N(F_N)$, for* ***some*** *sequence $(F_N)_{N\ge1}$ converging to G in $L^2(\Pi^d)$.*

5. Scaling Limit

In this section, following the ideas of [6], we will combine all previous results about Γ-convergence to obtain the hydrodynamical limit of exclusion process. The central structure of the proof is the usual one for convergence of stochastic processes. First we prove tightness of the sequence of processes π^N and then we show that all limit points are concentrated on weak solution of the hydrodynamic equation (3.1). Uniqueness of such solutions, proved in subsection (5.3), concludes the proof.

To avoid complications about to deal with π^N directly, the strategy above will be done for another process $\hat{\pi}^N$, which is called in the literature by the *corrected* empirical measure, and is close to π^N, carrying out the results to this last one.

5.1. **The corrected empirical measure.** Let's proceed to rigoursly define the so-called empirical measure. We begin by citing a general fact about minimizers of quadratic forms in a hilbert space X:

Proposition 5.1. *Let $F : X \to [0, +\infty]$ be a quadratic form, let A be the corresponding operator on $V = \overline{D(F)}$, and let $P : X \to V$ be the orthogonal projection onto V. For every $x, f \in X$, the following conditions are equivalent:*

(a) *$x \in D(A)$ and $Ax = Pf$;*
(b) *$F(y) \ge F(x) + 2\langle f, y - x\rangle$;*
(c) *x is a miminum point in X of the functional $G(y) = F(y) - 2\langle f, y - x\rangle$.*

Proof. E.g. [10], page 141, proposition 12.12. □

Coming back to our situation, define the functionals

$$\begin{aligned} \mathcal{E}_N^G(F) &= \langle(\lambda - \mathbb{L}_N)S_N F, S_N F\rangle_N - 2\langle S_N F, S_N G\rangle_N \\ \mathcal{E}^G(F) &= \langle(\lambda - \mathcal{L}_\Lambda)F, F\rangle - 2\langle F, G\rangle \end{aligned}$$

Recall the definitions of $\mathcal{E}_N$ and $\mathcal{E}$. By $\mathcal{E}_N \stackrel{\Gamma}{\to} \mathcal{E}$ proved in section (4), proposition (4.2) and proposition (convergência gamma + uniforme $\Rightarrow$ gamma), we achieve $\mathcal{E}_N^G \stackrel{\Gamma}{\to} \mathcal{E}^G$. This convergence, plus the coerciviness also proved in section (4), implies existence of a sequence of minimizers F_N of $\mathcal{E}_N^G$ converging in L^2 to the minimizer F of $\mathcal{E}^G$. By the proposition (5.1) above about minimizers, we have the two equations $(\lambda - \mathbb{L}_N)S_N F_N = S_N G$ and $(\lambda - \mathcal{L}_\Lambda)F = G$. Using this equations and again (4.2), it implicates $\mathcal{E}_N(F) = \langle S_N G, S_N F\rangle \to \langle G, F\rangle = \mathcal{E}(F)$. Now we are ready to define the corrected empirical measure $\hat{\pi}_t^N$. Let $F \in \mathcal{D}_\Lambda$ and define G by $G = (\lambda - \mathcal{L}_\Lambda)F$. Define F_N as the minimizer of $\mathcal{E}_N^G$ (notice that in this way $S_N F_N$ is unique). Then we define

$$\hat{\pi}_t^N(F) = \frac{1}{N^d}\sum_{x\in T^N} \eta_t^N(x) S_N F_N(x).$$

Remark: In spite of the name, $\hat{\pi}_t^N$ is not clear if well defined as a measure in Π^d.

5.2. **Tightness and proof of hydrodynamic limit.** Let $F \in \mathcal{D}_\Lambda$ and F_N the sequence of minimizers defined as before. As already seen, $F_N \stackrel{L^2}{\to} F$, which implies, by (4.2),

$$\lim_{N\to\infty} \frac{1}{N^d}\sum_{x\in T^N} |S_N F_N(x) - S_N F(x)| = 0.$$

And from this, we get

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T} |\pi_t^N(F) - \hat{\pi}_t^N(F)| > \epsilon\right) = 0.$$

Therefore, $\{\hat{\pi}_t^N(F), 0 \le t \le T\}_{N\ge 0}$ is tight in $D([0,T],\mathbb{R})$ if, and only if, $\{\pi_t^N(F), 0 \le t \le T\}_{N\ge 0}$ is. So, if we show the tightness of $\{\hat{\pi}_t^N(F), 0 \le t \le T\}_{N\ge 0}$, the density of $\mathcal{D}_\Lambda$ will guarantee the tightness of $\{\pi_t^N, 0 \le t \le T\}_{N\ge 0}$ in $D([0,T], \mathcal{M}_+^1)$. For references, see [5].

By Dynkyn's formula and a simple calculation,

$$M_t^N(F) = \hat{\pi}_t^N(F) - \hat{\pi}_0^N(F) - \int_0^t \frac{1}{N^d}\sum_{x\in T^N} \eta_s^N(x)\mathbb{L}_N S_N F_N(x) ds \tag{5.1}$$

is a martingale. His quadratic variation is given by

$$\langle M^N(F)\rangle_t = \int_0^t \frac{1}{N^d}\sum_{x,y\in T_N} (\eta_s^N(y) - \eta_s^N(x))^2 \xi_{x,y}^N (S_N F_N(y) - S_N F_N(y))^2 ds.$$

In particular, we have $\langle M^N(F)\rangle_t \le \frac{t}{N^d}\mathcal{E}_N(F_N)$. Because $\mathcal{E}_N(F_N) \to \mathcal{E}(F)$, the martingale converges to zero in L^2, and, by Doob's inequality, it is tight (in the uniform topology, thus in the Skorohod topology). On the other hand, by using the two equations $(\lambda - \mathbb{L}_N)S_N F_N = S_N G$ and $(\lambda - \mathcal{L}_\Lambda)F = G$, we can rewrite the integral term in (5.1) as

$$\int_0^t \frac{1}{N^d}\sum_{x\in T^N} \eta_s^N(x)\Big[S_N(\mathcal{L}_\Lambda F)(x) + \lambda\, S_N(F - F_N)(x)\Big] ds$$

$$= \int_0^t \pi_s^N\Big(S_N(\mathcal{L}_\Lambda F) + \lambda\, S_N(F - F_N)\Big) ds.$$

It is easy to see that $|\pi_s^N(H)| \leq \int_{\Pi^d} |H(u)|du$, which together with the convergence of minimizers $F_N \stackrel{L^2}{\to} F$ (notice that the norm of S_N as a projection is uniformly limited), yields the bounded variation of the integral term, uniformly in N. By [5], tightness follows at once, and we get as well the convergence result: $\forall \varepsilon > 0$, $\forall F \in \mathcal{D}_\Lambda$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\leq t\leq T}\Big|\pi_t^N(F) - \pi_0^N(F) - \int_0^t \pi_s^N(\mathcal{U}_\Lambda F)ds\Big| > \varepsilon\Big] = 0.$$

Let $\pi_t, 0 \leq t \leq T$, be any limit point of $\{\pi_t^N, 0 \leq t \leq T\}_{N\geq 0}$. Then, by the convergence above, $\pi_t, 0 \leq t \leq T$ satisfies the identity

$$\pi_t(F) - \pi_0(F) - \int_0^t \pi_s(\mathcal{U}_\Lambda F)ds = 0, \;\; 0 \leq t \leq T,$$

for any function $F \in \mathcal{D}_\Lambda$. The uniqueness of bounded solutions of such equations proved in subsection (5.3) finishes the hydrodynamic limit.

5.3. **Uniqueness of weak solutions.** In this subsection, we are to prove the uniqueness of weak solutions of (3.1), which strategy follows [4]. It suffices to check that the only solution of (3.1) with $\gamma \equiv 0$ is $\rho \equiv 0$, because of linearity of $\mathcal{L}_\Lambda$. Let $\rho : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ be a weak solution of the parabolic differential equation

$$\begin{cases} \partial_t \rho \;=\; \mathcal{U}_\Lambda \rho \\ \rho(0,\cdot) \;=\; 0. \end{cases}$$

Then,

$$\langle \rho_t, H\rangle \;=\; \int_0^t \langle \rho_s, \mathcal{U}_\Lambda H\rangle \, ds \;, \tag{5.2}$$

for all functions H in $\mathcal{D}_\Lambda$ and all $t > 0$. From (Teorema sobre autovalores de $\mathcal{U}_\Lambda$) the operator $-\mathcal{U}_\Lambda$ has countable eigenvalues $\{\lambda_n : n \geq 0\}$ and eigenvectors $\{f_n\}$. All eigenvalues have finite multiplicity, $0 = \lambda_0 \leq \lambda_1 \leq \cdots$, and $\lim_{n\to\infty} \lambda_n = \infty$. The eigenvectors $\{f_n\}$ form a complete orthonormal system in the $L^2(\mathbb{T})$. Define

$$R(t) = \sum_{n\in\mathbb{N}} \frac{1}{n^2(1+\lambda_n)} \langle \rho_t, f_n\rangle^2,$$

for all $t > 0$ and $R(0) = 0$. $R(t)$ is well defined because ρ_t belongs to $L^2(\mathbb{T})$ and $\{f_n\}$ is a complete orthonormal system in the $L^2(\mathbb{T})$. Since ρ satisfy (5.2), we have that $\frac{d}{dt}\langle \rho_t, f_n\rangle^2 = -2\lambda_n \langle \rho_t, f_n\rangle^2$. Then

$$(\frac{d}{dt}R)(t) = -\sum_{n\in\mathbb{N}} \frac{2\lambda_n}{n^2(1+\lambda_n)} \langle \rho_t, f_n\rangle^2,$$

because $\sum_{n\leq N} \frac{-2\lambda_n}{n^2(1+\lambda_n)} \langle \rho_t, f_n\rangle^2$ converges uniformly to $\sum_{n\in\mathbb{N}} \frac{-2\lambda_n}{n^2(1+\lambda_n)} \langle \rho_t, f_n\rangle^2$, when N increases to infinity. Thus $R(t) \geq 0$ and $(\frac{d}{dt}R)(t) \leq 0$, for all $t \geq 0$ and $R(0) = 0$. From this, we obtain $R(t) = 0$ for all $t \geq 0$. Then $\langle \rho_t, \rho_t\rangle = 0$, for all $t \geq 0$, which implies $\rho \equiv 0$.

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