A VERSION OF THE RANDOM DIRECTED FOREST AND ITS CONVERGENCE TO THE BROWNIAN WEB.

GLAUCO VALLE AND LEONEL ZUAZNÁBAR.

Abstract. Several authors have studied convergence in distribution to the Brownian web under diffusive scaling of Markovian random walks. In a paper by R. Roy, K. Saha and A. Sarkar, convergence to the Brownian web is proved for a system of coalescing random paths - the Random Directed Forest- which are not Markovian. Paths in the Random Directed Forest do not cross each other before coalescence. Here we study a generalization of the non-Markovian Random Directed Forest where paths can cross each other and prove convergence to the Brownian web. This provides an example of how the techniques to prove convergence to the Brownian web for systems allowing crossings can be applied to non-Markovian systems.

1. Introduction.

Several authors have studied convergence in distribution to the Brownian web for different processes, for instance [BMSV06],[CV14],[FFW14],[FINR04],[FVV15],[RSS16] to mention some works. The aim at most of these papers is the understanding of the universality class associated to the Brownian web. It was formaly introduced in [FINR04] where only nearest neighbor simple symmetric random walks have been considered. This was a breakthrough because it was an important question in probability theory about how to characterize properly the convergence of systems of coalescing random walks which started to be studied by Arratia in [Arr81]. From [FINR04] the question about the universality class for the Brownian web arises as important one since many important systems of coalescing random paths related to applications of probability theory are more complicated, for instance they may have long range dependence, they are not necessarily independent before coalescence and they are not necessarily Markovian, see for instance the Poisson Tree in [FFW14], the Drainage Network Model studied in [CFD09] and [CV14]; the Random Directed Forest studied in [RSS16] or the Direct Spaning Forest in [BB07], where the authors made a conjecture about the convergence to a transformation of the Brownian web in its Remark 4.9 that was proved for a similar system in [FVV15]. You can find a review to the Brownian web, and how they arise in the scaling limits of various one-dimensional models in [SSS15].

In [RSS16] the authors study the Random Directed Forest which is a system of coalescing space and time random paths on $\mathbb{Z}^2$ as we now describe. Suppose that the first coordinate of a point in $\mathbb{Z}^2$ represents space and the second one time. We start a space/time random path in each point of $\mathbb{Z}^2$. The path starting at u in $\mathbb{Z}^2$ evolves as follows: every point in $\mathbb{Z}^2$ is open with some probability p or closed with $1-p$ independently of each other. We say that a point $v = (\tilde{x}, \tilde{t}) \in \mathbb{Z}^2$ is above $u = (x, t)$ if $\tilde{t} > t$. If the path is at space/time position (v, t) then it jumps to the nearest open point in the L_1 norm above (v, t) if this nearest open point is unique. If it is not unique then the a choice is made uniformly to decide where the path has to jump to (see the Figure 1). Note that two paths cannot cross each other and after one step it is possible to know something about the future, that is to say, maybe we know if some points above the current position of the path are open or closed. That is why we get a system of non-Markovian random paths. R. Roy, K. Saha and A.

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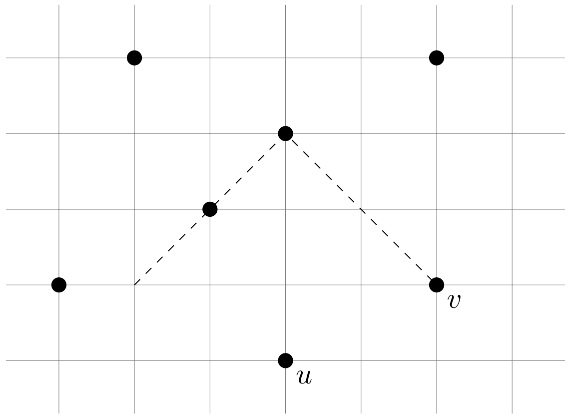

FIGURE 1. Open points in $\mathbb{Z}^2$ are marked by black dots. Note that the closest open points above u in the L_1 distance are those connected by the dashed line. Hence the path starting at u moves to one of these points connected by the dashed line chosen uniformly among them; for instance it could be v.

Sarkar in [RSS16] proved that under diffusive scaling, the closure of linearly interpolation trajectory induced by each discrete random path, in some space where the Brownian web is defined, converges in distribution to the Brownian web. Our initial aim was to to consider a generalization of the random directed forest that allows crossings before coalescence analogous to the generalized drainage models studied in [CV14]. This could be made if we do not impose the necessity that the jump should be made to the nearest open above position. Although we get a well defined system, we were not able to prove convergence to the Brownian web in this case. That problem here was to build a regeneration structure similar to that presented in [RSS16] which is the way around the non-Markovianity.

We will define a model which is slightly different from the Random Directed Forest and consider a generalization of it that allows crossing before coalescence. Suppose now that in each point u in $\mathbb{Z}^2$ we have a random variable W_u such that $\{W_u; u \in \mathbb{Z}^2\}$ is an i.i.d. family of random variables in the set of positive integers. We will call the k-th level of some $u = (u(1), u(2))$ in $\mathbb{Z}^2$ the following set

$$L(u,k) := \big\{v = (v(1), v(2)) \in \mathbb{Z}^2; v(2) > u(2) \text{ and } ||v-u||_1 = k\big\},$$

where for any $u \in \mathbb{Z}^2$, $||u||_1 := |u(1)| + |u(2)|$. A level $L(u,k)$ is called open if it has at least one open point. Now consider that the path not necessarily move to the nearest open point above him but the highest open point in the W_u-th open level. If the path has two options to jump to it makes an uniform choice to decide. See the Figure 2 for some example.

As the Directed Random Forest we now have a system of non-Markovian walks but in this case the paths can cross each other. Our goal is to prove convergence in distribution to the Brownian web under diffusive scaling for the closure of the system of linearly interpolated paths, see Theorem 2.3.

In the next section, Section 2, we are going to define formally the variation of the Random Directed Forest and state the convergence to the Brownian web. By the end of the same Section 2 we shall explain how the rest of the paper is divided in accordance with the steps that should be taken to prove the convergence result.

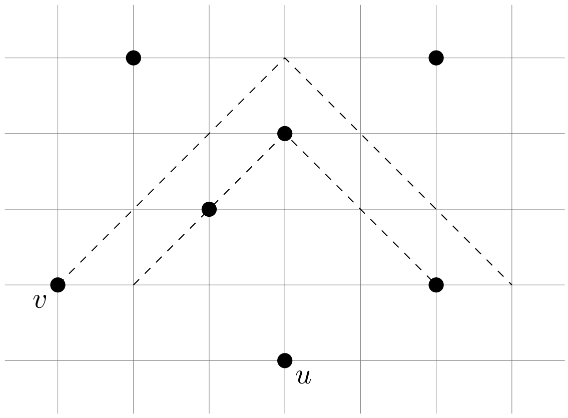

FIGURE 2. Note that points connected by the dashed lines are the first and the second open levels of u. If $W_u = 2$, for instance, the path starting in u jumps to v.

2. THE PROCESS AND THE BROWNIAN WEB.

Let us define formally the process described in the previous section. First let us fix some notation that will be used in the paper. We will denote by $\mathbb{Z}_+ := \{0,1,2,3,\dots\}, \mathbb{Z}_- := \{0,-1,-2,-3,\dots\}$ and $\mathbb{N} := \{1,2,3,\dots\}$. Consider the following random variables:

(i) Let $(W_u)_{u\in\mathbb{Z}^2}$ be a family of i.i.d. random variables with finite support on $\mathbb{N}$ such that $\mathbb{P}[W_u = 1] > 0$. Denote by $\mathbb{P}_W$ the induced probability on $\mathbb{N}^{\mathbb{Z}^2}$.

(ii) Let $\{U_v; v \in \mathbb{Z}^2\}$ be a family of i.i.d. Uniform random variables in $(0,1)$. Denote by $\mathbb{P}_U$ the induced probability on $(0,1)^{\mathbb{Z}^2}$.

We suppose that the two families above are independent of each other and thus have a joint distribution given by the product probability $\mathbb{P} := \mathbb{P}_W \times \mathbb{P}_U$ on the space $\mathbb{N}^{\mathbb{Z}^2} \times (0,1)^{\mathbb{Z}^2}$.

Now fix some $p \in (0,1)$. We write $u = (u(1), u(2))$ for $u \in \mathbb{Z}^2$ and call open the points in $V := \{u \in \mathbb{Z}^2; U_u < p\}$ and closed the points in $\mathbb{Z}^2 \setminus V$. For $u \in \mathbb{Z}^2$ and $k \in \mathbb{N}$ let us define its k-th level as

$$L(u,k) := \{v \in \mathbb{Z}^2; v(2) > u(2) \text{ and } ||v-u||_1 = k\}.$$

We will denote the index of the r-th open level of u by $h(u,r)$, i.e.

$$h(u,r) := \inf\Big\{k \geq 1; \sum_{j=1}^{k} {}_{\{L(u,j)\cap V \neq \emptyset\}} = r\Big\}.$$

For $u \in \mathbb{Z}^2$ denote by $X(u)$ the unique (almost surely) point in $L(u, h(u, W_u)) \cap V$ such that for every $w \in L(u, h(u, W_u)) \cap V$ either $X(u)$ is above w or $U_{X(u)} > U_w$. Let us define the sequence $\{X_n(u)\}_{n\geq 0}$ as,

$$X_0(u) := u \text{ and } X_n(u) := X(X_{n-1}(u)) \text{ for } n \geq 1.$$

Now define $\pi^u : [v(2), \infty] \to [-\infty, \infty]$ as $\pi^u(X_n(u)(2)) := X_n(u)(1)$, linearly interpolated in $[X_n(u)(2), X_{n+1}(u)(2)]$ and $\pi^u(\infty) = \infty$. Let us denote the set of paths by

$$\mathcal{X} := \big\{(\pi^v, v(2)); v \in \mathbb{Z}^2\big\}. \tag{2.1}$$

The system $\mathcal{X}$ is the modified Random Directed Forest which is the main object of study in this paper. From now on we call it the Generalized Random Directed Forest (GRDF).

We are interested in the diffusive rescaled GRDF. So let $\gamma > 0$ and $\sigma > 0$ be some fixed normalizing constants to be determined latter, $u \in \mathbb{Z}^2$ and $n \in \mathbb{N}$. Let us define $\pi_n^u(t) := \frac{\pi^u(n^2\gamma t)}{n\sigma}$ for $t \in [0,\infty)$, $\pi_n^u(\infty) =$

∞ and

$$\mathcal{X}_n := \big\{(\pi_n^v, v(2)); v \in \mathbb{Z}^2\big\}. \tag{2.2}$$

The system of coalescing paths $\mathcal{X}_n$ is the rescaled GRDF and our aim is to prove that its closure converges to the Brownian web as $n \to \infty$.

So now let us introduce the Brownian web. As in [FINR04] take $(\bar{\mathbb{R}}^2, \rho)$ a completion of $\mathbb{R}^2$ under the metric ρ defined as

$$\rho\big((x_1,t_1),(x_2,t_2)\big) := \Big|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\Big| \vee \big|\tanh(t_1) - \tanh(t_2)\big|.$$

We may think $\bar{\mathbb{R}}^2$ as the image of $[-\infty,\infty] \times [-\infty,\infty]$ under the mapping

$$(x,t) \to \big(\Phi(x,t), \Psi(t)\big) := \Big(\frac{\tanh(x)}{1+t}, \tanh(t)\Big).$$

For $t_0 \in [-\infty,\infty]$, let $C[t_0]$ be the set of functions from $[t_0,\infty]$ to $[-\infty,\infty]$ such that $\Phi\big(f(t),t\big)$ is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty,\infty]} C[t_0] \times \{t_0\}.$$

For (f,t_0) in Π, let us denote $\widehat{f}$ the function that extends f to all $[-\infty,\infty]$ by setting it equal to $f(t_0)$ for $t \leq_0$. Take

$$d\big((f_1,t_1),(f_2,t_2)\big) = \Big(\sup_t |\Phi(\widehat{f}_1(t),t) - \Phi(\widehat{f}_2(t),t)|\Big) \vee |\Psi(t_1) - \Psi(t_2)|.$$

Let now $\mathcal{H}$ denote the set of compact subset of (Π, d) with the Hausdorff metric $d_{\mathcal{H}}$,

$$d_{\mathcal{H}}(K_1,K_2) := \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1,g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1,g_2),$$

for K_1, K_2 non-empty sets in $\mathcal{H}$. $\mathcal{F}_{\mathcal{H}}$ is the Borel σ-field induced by $(\mathcal{H}, d_{\mathcal{H}})$.

The existence of a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$- valued random variable, called as the Brownian web, with the convergence properties that we had mentioned, was proved in the Theorem 2.1 in [FINR04].

Theorem 2.1. *There exists a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})-$ valued random variable $\mathcal{W}$ whose distribution is uniquely determined by the following three properties:*

- (i) *For any deterministic point (x,t) in $\mathbb{R}^2$ there exists almost surely an unique path $\mathcal{W}_{x,t}$ starting from (x,t).*
- (ii) *For any deterministic $n, (x_1,t_1), \dots, (x_n,t_n)$ the joint distribution of $\mathcal{W}_{(x_1,t_1)}, \dots, \mathcal{W}_{(x_n,t_n)}$ is that of coalescing Brownian motions.*
- (iii) *For any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure in $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ of $\{\mathcal{W}_{x,t} : (x,t) \in \mathcal{D}\}$.*

The next Theorem 2.2 is a criteria of convergence to the Brownian web. This theorem is a variation of the Theorem 2.2 proved in [FINR04] which can be found as the Theorem 1.4 in [NRS05]. These theorems (Theorem 2.2 in [FINR04] and Theorem 1.4 in [NRS05]) has been the principal tools to prove the convergence to the Brownian web for many different kind of coalescing system.

Theorem 2.2. *Let $\{\mathcal{Y}_n\}_{n\geq 1}$ be a sequence of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued r.v. We have that $\{\mathcal{Y}_n\}_{n\geq 1}$ converges to the Brownian web if the following conditions are satisfied:*

- (I) *There exist some deterministic countable dense subset of $\mathbb{R}^2$, let us called D, and $\theta_n^y \in \mathcal{Y}_n$ for any $y \in D$ satisfying: for any deterministic $y_1, \dots, y_m \in D, \theta_n^{y_1}, \dots, \theta_m^{y_m}$ converge in distribution as $n \to \infty$ to coalescing Brownian motions starting in $y_1, \dots, y_m$.*

(*B*) $\forall \beta > 0, \limsup_n \sup_{t>\beta} \sup_{t_0,a\in\mathbb{R}} \mathbb{P}\big[|\eta_{\mathcal{Y}_n}(t_0,t,a-\epsilon,a+\epsilon)| > 1\big] \to 0,$ *as* $\epsilon \to 0^+$, *where* $\eta_{\mathcal{Y}_n}(t_0,t,a,b) = \big\{y \in \mathbb{R} \times \{t_0+t\}$; *are touched by paths which also touch some point in* $[a,b] \times \{t_0\}\big\}$.

(*E*) *For some* $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$*-valued r.v.* Y *and* $t > 0$ *take* Y^{t^-} *as the subset of paths in* Y *which start before or at time* t. *If* Z_{t_0} *is the subsequential limit of* $\{\mathcal{Y}_n^{t_0^-}\}_{n\geq 1}$ *for any* t_0 *in* $\mathbb{R}$, *then for all* t, a, b *in* $\mathbb{R}$ *with* $t > 0$ *and* $a < b$ *we get*

$$\mathbb{E}\big[|\eta_{Z_{t_0}}(t_0,t,a,b)|\big] \leq 1 + \frac{b-a}{\sqrt{\pi t}}.$$

(*T*) *Let* $\Lambda_{L,T} := [-L,L] \times [-T,T] \subset \mathbb{R}^2$ *and for* $(x_0,t_0) \in \mathbb{R}^2$ *and* $\rho, t > 0, R(x_0,t_0;\rho,t) := [x_0-\rho, x_0+\rho] \times [t_0, t_0+t] \subset \mathbb{R}^2$. *For* $K \in \mathcal{H}$ *define* $A_{\mathcal{K}}(x_0,t_0;\rho,t)$ *to be the event that* K *contains a path touching both* $R(x_0,t_0;\rho,t)$ *and the right or the left boundary of the rectangle* $R(x_0,t_0;20\rho,4t)$. *Then for every* $\rho, L, T \in (0,\infty)$

$$\frac{1}{t} \limsup_{n\to\infty} \sup_{(x_0,t_0)\in\Lambda_{L,T}} \mathbb{P}\Big[A_{\mathcal{Y}_n}(x_0,t_0;\rho,t)\Big] \to 0 \text{ as } t \to 0^+.$$

The main result in this paper is the convergence of the GRFD to the Brownian web under diffusive scaling stated below.

Theorem 2.3. *There exist positive constants* γ *and* σ *such that* $\overline{\mathcal{X}}_n$, *the closure of* $\mathcal{X}_n$ *in* (Π, d), *converges in distribution to the Brownian web as* n *goes to infinity.*

The rest of the paper is devoted to the proof of Theorem 2.3 and we end this section explaining how it divided. In Section 3 we introduce regeneration times where the random paths in the GRDF have no information about the future, this yields a Markovian structure we can rely on. In Section 4 we prove a central estimate related to the tail probability of the coalescing time of two random paths of the GRDF. The results from both Sections 3 and 4 will be essential for the rest of the paper. In sections 5, 6, 7 and 8 we prove respectively conditions I, B, E and T. Finally we end the paper with and appendix where we show that $\overline{\mathcal{X}}_n$ is a well defined random elements of $(\mathcal{H}, d_{\mathcal{H}})$. In the appendix we also prove that every path in $\overline{\mathcal{X}}_n$ from any time $t \in \mathbb{R}$ coincide with some path in $\mathcal{X}_n$ and this result allows us to prove conditions B, E and T in Theorem 2.2 working with $\mathcal{X}_n$ instead of $\overline{\mathcal{X}}_n$.

3. Renewal times

In this section we prove the existence of regeneration times where the random paths in the GRDF have no information about the future. The idea of using regeneration times came from [RSS16] and is fundamental since the paths seen at these times have the Markov property. However we are not able to get the existence as they did it, because in our case the paths get into regions which has been observed before, something that the Random Directed Forest do not do and is used in the proof given in [RSS16]. So we follow a different approach here. The hypothesis impose to the r.v $\{W_u; u \in Z^2\}$ will be needed in the proof.

As in [RSS16] let us denote by $\Delta_k(u)$, for $k \in \mathbb{Z}_+$ and $u \in \mathbb{Z}^2$, the set of points above $X_k(u)$ whose configuration is already known; i.e. $\Delta_0(u) = \emptyset$ and for $k \geq 1$,

$$\Delta_k(u) := \Big[\Delta_{k-1}(u) \cup \big\{v \in \mathbb{Z}^2; ||v - X_{k-1}(u)||_1 \leq ||X_k(u) - X_{k-1}(u)||_1\big\}\Big] \cap \big\{v \in \mathbb{Z}^2; v(2) > X_k(u)(2)\big\}. \tag{3.1}$$

See Figure 3 as an example.

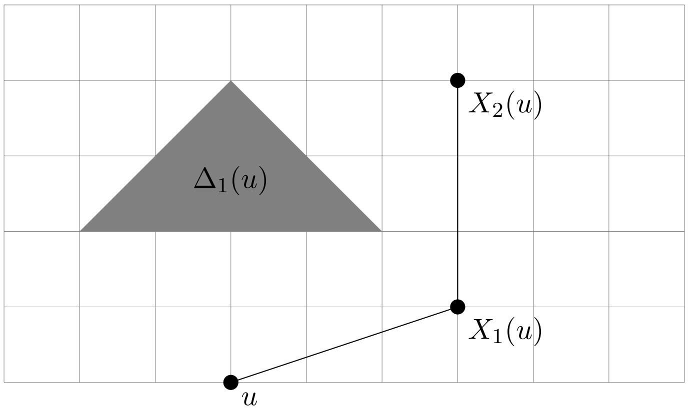

FIGURE 3. Example for the dependence region $\Delta_1(u) = \emptyset$. Note that in this example $\Delta_2(u) = \emptyset$.

For any random variable $\tau(u)$ which satisfies $\Delta_{\tau(u)}(u) = \emptyset$ we call $X_{\tau(u)}(u)(2)$ a renewal time for the random path $\{X_k(u); k \geq 1\}$. Note that $\tau(u)$ is not necessarily the first k such that $\Delta_k(u) = \emptyset$. The fact that the paths do not jump necessarily to first open level above it do not allow us to use the approach in [RSS16] to verify existence and moment conditions of renewal times. The main result of this section is the following:

Proposition 3.1. *Let $u_1, \dots, u_m$ be points in $\mathbb{Z}^2$ at the same time level, i.e with equal second component. Then there exist random variables T, Z and $\tau(u_i)$ for $i = 1 \dots, m$ such that $T \leq Z$ and*

(i) *$\Delta_{\tau(u_i)}(u_i) = \emptyset$ and $X_{\tau(u_1)}(u_1)(2) = X_{\tau(u_i)}(u_i)(2)$ for all $i = 1, \dots, m$.*
(ii) *Taking $T := X_{\tau(u_1)}(u_1)(2)$ we have that its distribution depends on m but not on $u_1, \dots, u_m$. For all $k \geq 1$ we get $\mathbb{E}[T^k] < \infty$. Note that $\pi^{u_i}(T) = X_{\tau(u_i)}(u_i)(1)$.*
(iii) *For all $i = 1, \dots, m$ we have $\sup_{0 \leq t \leq T} |\pi^{u_i}(t) - u_i(1)| \leq Z$ and its distribution depends on m but not on $u_1, \dots, u_m$. Also for all $k \geq 1$ we get $\mathbb{E}[Z^k] < \infty$.*

Proof. Without lost of generality we can assume that $u_1(2) = \dots = u_m(2) = 0$. Let K be a constant such that $\sum_{i=1}^{K} \mathbb{P}[W_{(0,0)} = i] = 1$. This constant exists because the finite support hypothesis. For $u \in \mathbb{Z}^2$ let us define the following event

$$E(u) := \big\{(u(1), u(2) + j) \text{ is open }; j = 1, \dots, K\big\} \cap \big\{W_{(u(1),u(2)+j)} = 1; j = 1, \dots, K - 1\big\}.$$

Note that on $E(u)$ the path that start in u after some steps arrive in $(u(1), u(2) + K)$ and then he knows nothing about the point above. Now take $\{\widehat{E}_{1,j}; 1 \leq j \leq m\}$ independent events such that $\mathbb{P}[\widehat{E}_{1,j}] = \mathbb{P}[E((0, 0))]$ for $j = 1, \dots, m$ and independents of the process too. Let us define

$$D_1 := \{j \in \{1, \dots, m\}; u_j(1) = u_i(1) \text{ for some } 1 \leq i < j\} \text{ and } E_1 := \Big[\bigcap_{j\in\{1,\dots,m\}\setminus D_2} E(u_j) \cap \Big[\bigcap_{j\in D_1} \widehat{E}_{1,j}\Big].$$

Then on E_1 the paths that start in $u_1, \dots, u_m$ after some $\tau(u_1), \dots, \tau(u_m)$ steps, respectively, they will arrive in the points $(u_1(1), u_1(2) + K), \dots, (u_m(1), u_m(2) + K)$ and $\Delta_{\tau(u_j)} = \emptyset$ for $j = 1, \dots, m$. We will find a sequence of independent events $\{E_n\}_{n\geq1}$ with the same probability of success of E_1 such that if E_n occurs for some n, we have a joint renewal time for the paths that start in $u_1, \dots, u_m$. To do this let us make some definitions. Define the following upper bound to high of $\Delta_1(u)$,

$$H(u) := \inf \Big\{n \geq 1; \sum_{j=1}^{n} \ _{\{(u(1),u(2)+j) \text{ is open }\}} = K\Big\}. \tag{3.2}$$

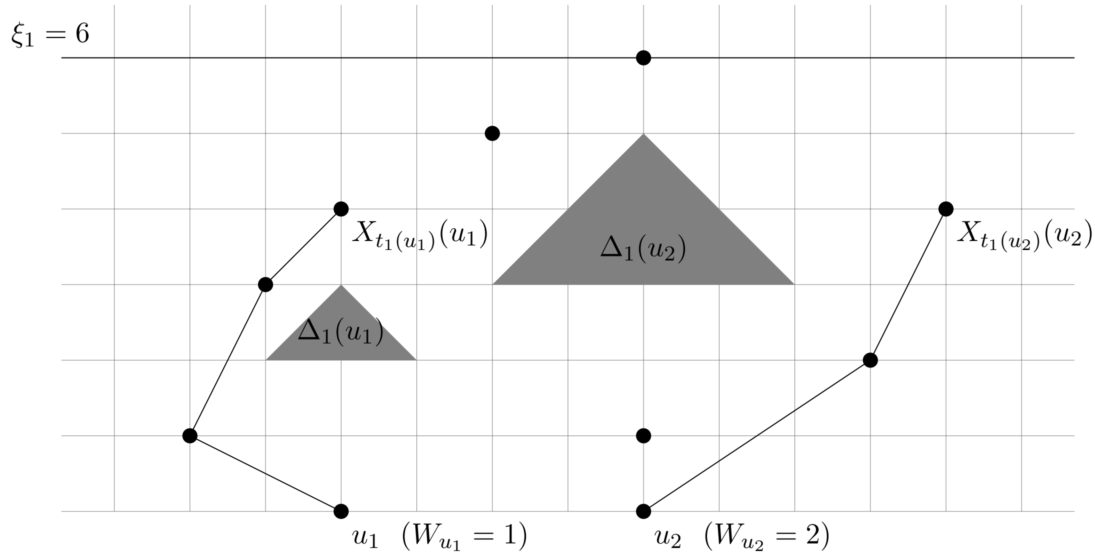

FIGURE 4. In the picture above we consider a realization of the random paths in the GRDF starting at u_1 and u_2. In this case $\xi_1 = 6$ and one can see that the dependence region generated by the first for both paths are below $u_1(2) + \xi_1$. Moreover note that $X_{t_1(u_1)}(u_1)$ and $X_{t_1(u_2)}(u_2)$ are the last points visited by the paths starting respectively at u_1 and u_2 before time $u_1(2) + \xi_1$.

Take $\{\widehat{H}_{1,j}; 1 \leq j \leq m\}$ i.i.d. random variable independent of the model and with the same distribution of $H((0,0))$. Now define

$$\xi_1 := \Big[\max_{j \in \{1,\dots,m\} \setminus D_1} H(u_j)\Big] \vee \Big[\max_{j \in D_1} \widehat{H}_{1,j}\Big].$$

Now we will move each path until the first time that they need to observe over ξ_1 to decide where to jump. These times could be defined as

$$t_1(u_j) := \inf\Big\{n \geq 1; X_n(u)(2) = \xi_1 \text{ or } \sum_{i=1}^{\xi_1 - X_n(u)(2)} \mathbb{1}_{\{L(u,i) \text{ is open }\}} < W_{X_n(u)}\Big\},$$

for all $1 \leq j \leq m$. Note that to define $t_1(u_j)$ we do not need to see the points above ξ_1. To help the understanding of the notation see Figure 4.

Now take $\{\widehat{E}_{n,j}; 1 \leq j \leq m, n \geq 2\}$ a family of independent event and independent of the model too, such that $\mathbb{P}[\widehat{E}_{n,j}] = \mathbb{P}[E((0,0))]$ for all $1 \leq j \leq m$ and $n \geq 2$. Also take $\{\widehat{H}_{n,j}; 1 \leq j \leq m, n \geq 2\}$ an i.i.d. family of random variable independent of the model and with the same distribution of $H((0,0))$. Getting defined $E_1, \dots, E_m$ and $\xi_1, \dots, \xi_n$ we can define E_{n+1} and ξ_{n+1} as follows. First take

$$t_n(u_j) := \inf\Big\{k \geq 1; X_k(u)(2) = \xi_1 + \dots \xi_n \text{ or } W_{X_k(u)} > \sum_{i=1}^{\xi_1 + \dots \xi_k - X_k(u)(2)} \mathbb{1}_{\{L(u,i) \text{ is open }\}}\Big\},$$

for all $1 \leq j \leq m$. Define

$$D_{n+1} := \Big\{j \in \{1, \dots, m\}; X_{t_n(u_j)}(u_j)(1) = X_{t_n(u_i)}(u_i)(1) \text{ for some } 1 \leq i < j\Big\}$$

and

$$E_{n+1} := \Big[\bigcap_{i\in\{1,\dots,m\}\setminus D_{n+1}} E\big((X_{t_n(u_i)}(u_i)(1), \xi_1+\cdots+\xi_n)\big)\Big] \cap \Big[\bigcap_{j\in D_{n+1}} \widehat{E}_{n,j} \Big]$$
$$\xi_{n+1} := \Big[\max_{j\in\{1,\dots,m\}\setminus D_{n+1}} H\big((X_{t_n(u_j)}(u_j)(1), \xi_1+\cdots+\xi_n)\big)\Big] \vee \Big[\max_{j\in D_{n+1}} \widehat{H}_{n,j}\Big].$$

Note that $\{\xi_n\}_{n\geq 1}$ is an i.d.d. sequence and the distribution of ξ_n does not depend on $u_1(1),\dots,u_m(1)$. Also the probability of success of the event E_n does not depend on $u_1(1),\dots,u_m(1)$ and it is equal to $P[E_1]$. See that if E_n happens fore some n then we get the renewals for the paths. Now defining the geometric random variable $M := \inf\{n\geq 1; \ _{E_n} = 1\}$ and $\tau(u_j) = t_M(u_j)$ we get $\Delta_{\tau(u_j)}(u_j) = \emptyset$ for all $1\leq j\leq m$. Defining $T := \sum_{i=1}^M \xi_i$ we get that $T = X_{\tau(u_j)}(u_j)(2)$ for all $1\leq j\leq m$ and applying the Lemma A.1, $\mathbb{E}[T^l]<\infty$ for all $l\in\mathbb{N}$. Note that the distribution of T does not depend on $u_1(1),\dots,u_m(1)$. Now see that for $j = 1\dots,m$ by the construction of $\{\xi_k; k\geq 1\}$ and $\{t_k(u_j); k\geq 1\}$ we have

$$\sup_{0\leq t\leq T} |\pi^{u_j}(t) - u_j(1)| \leq \sum_{k=1}^{M} (\xi_1+\cdots+\xi_k)^2 \leq M(\xi_1+\dots\xi_M)^2 \leq (\xi_1+\dots\xi_M)^3 := Z.$$

Using the Lemma A.1 we get $\mathbb{E}[Z^l]<\infty$ for all $l\geq 1$ and by construction the distribution of Z does not depends on $u_1,\dots,u_m$.

□

We can replicate recursively Proposition 3.1 to get:

Corollary 3.1. *Let $m\geq 1$ and $u_1,\dots,u_m$ be points in $\mathbb{Z}^2$ at the same level. Then there exist sequences of random variables $\{T_j\}_{j\geq1}$, $\{Z_j\}_{j\geq 1}$ and $\{\tau_j(u_i)\}_{j\geq 1}$ for $i=1,\dots m$ such that,*

(i) $\Delta_{\tau_j(u_i)}(u_i) = \emptyset$ *and* $X_{\tau_j(u_1)}(u_1)(2) = X_{\tau_j(u_i)}(u_i)(2)$ *for all* $i = 1,\dots,m$ *and* $j\geq 1$. *Moreover for every* $i = 1,\dots m$ *we have that* $\tau_j(u_i) - \tau_{j-1}(u_i)$, $j\geq 1$, *are i.i.d random variables.*

(ii) $X_{\tau_j(u_1)}(u_1)(2) = T_j$ *for* $j\geq 1$ *and its distribution depends on* m *but not on* $u_1,\dots,u_m$. *For all* $k,j\geq 1$ *we get that* $\mathbb{E}[T_j^k]<\infty$ *and for all* $j\geq 1, i = 1\dots m$ *we have that* $\pi^{u_i}(T_j) = X_{\tau_j(u_i)}(u_i)(1)$. *Moreover* $T_j - T_{j-1}$, $j\geq 1$, *are i.i.d random variables .*

(iii) $\sup_{T_{j-1}\leq t\leq T_j} |\pi^{u_i}(t) - \pi^{u_i}(T_{j-1})| \leq Z_j$ *for all* $i = 1,\dots,m$ *and* $j\geq 1$. *The random variables* Z_j, $j\geq 1$, *are i.i.d and their common distribution depends on* m *but not on* $u_1,\dots,u_m$. *For all* $k,j\geq 1$ *we get that* $\mathbb{E}[Z_j^k]<\infty$.

Fix points $u_1,\dots,u_m$ in $\mathbb{Z}^2$. To simplify suppose that u_1 is at the same time level or above u_i for every $i = 2,\dots m$. Then we can move each paths $\pi^{u_2},\dots,\pi^{u_m}$ up to the first time they need to see above $u_1(2)$ to move and after that use the same idea of Proposition 3.1 to obtain a similar renewal structure for paths in $\mathcal{X}$ that do not start necessarily at the same time level:

Corollary 3.2. *Let $m\geq 1$ and $u_1,\dots,u_m$ be points in $\mathbb{Z}^2$ such that u_1 is at the same time level or above u_i for every $i = 2,\dots m$. Then there exist sequences of random variables $\{T_j\}_{j\geq 1}$, $\{Z_j\}_{j\geq1}$ and $\{\tau_j(u_i)\}_{j\geq 1}$ for $i = 1,\dots m$ such that,*

(i) $\Delta_{\tau_j(u_1)}(u_1) = \Delta_{\tau_j(u_i)}(u_i) = \emptyset$ *and* $X_{\tau_j(u_1)}(u_1)(2) = X_{\tau_j(u_i)}(u_i)(2)$ *for all* $i = 1,\dots,m$ *and* $j\geq 1$. *Moreover we have that* $\tau_j(u_1) - \tau_{j-1}(u_1)$, $j\geq 1$, *are i.i.d random variables, and for every* $i = 2,\dots m$ *we have* $\tau_j(u_i) - \tau_{j-1}(u_i)$, $j\geq 2$, *are i.i.d random variables.*

(ii) $X_{\tau_j(u_i)}(u_i)(2) = T_j$ *for all* $i = 1,\dots m$ *and* $j\geq 1$. *Also for all* $j\geq 1, i = 1\dots m$ *we have that* $\pi^{u_i}(T_j) = X_{\tau_j(u_i)}(u_i)(1)$. *The distribution of* T_j *depends on* m *but not on* $u_1,\dots,u_m$ *and for all* $k,j\geq 1$ *we get that* $\mathbb{E}[T_j^k]<\infty$. *Moreover* $T_j - T_{j-1}$, $j\geq 1$, *are i.i.d random variables.*

(iii) $\sup_{T_{j-1}\le t\le T_j} |\pi^{u_1}(t) - \pi^{u_1}(T_{j-1})| \le Z_j$ *for all* $j \ge 1$, *and for every* $i = 2,\dots m$ *we have that* $\sup_{T_{j-1}\le t\le T_j} |\pi^{u_i}(t) - \pi^{u_i}(T_{j-1})| \le Z_j$ *for all* $j \ge 2$. *The random variables* Z_j, $j \ge 1$, *are i.i.d and their common distribution depends on* m *but not on* $u_1,\dots,u_m$. *For all* $k, j \ge 1$ *we get that* $\mathbb{E}[Z_j^k] < \infty$.

4. COALESCING TIMES

In this section we obtain an upper bound on the tail probability of the coalescence time of two paths in $\mathcal{X}$. This is a central estimate related to convergence to the Brownian web. Related to other processes see [CFD09],[CV14] and [RSS16] for instance. The main ideas used here to get the bound come from these three works, although it is not a straighforward application of the techniques used before. Here we have another important difference with the Random Directed Forest studied in [RSS16] because of the possibility of the paths to cross each other before coalescence. This property does not allow us to follow the proof in [CFD09] as done in [RSS16]. We will need the ideas used in [CV14] where the authors work with a system allowing crossing to obtain this upper bound. Here the hypothesis about the finite support of the r.v $\{W_u; u \in Z^2\}$ will be used to prove item $i)$ of Lemma 4.2 and Lemma 4.3 below. Although we believe that the bound on the coalescing time tail holds without the finite support hypothesis.

So the aim of this section is to prove the following result.

Proposition 4.1. *Let us define* $\nu := \inf\{t \ge 0; \pi^{(0,0)}(s) = \pi^{(0,1)}(s) \text{ for all } s \ge t\}$. *Then there exist a positive constant* $C > 0$ *such that*

$$\mathbb{P}[\nu > t] \le \frac{C}{\sqrt{t}}.$$

As an immediate consequence of Proposition 4.1 we have:

Corollary 4.1. *Let* $u = (0,0), v = (0,l)$ *and* $\nu(u,v) := \inf\{t \ge 0; \pi^u(s) = \pi^v(s) \text{ for all } s \ge t\}$. *Then there exist positive constant* C *such that*

$$\mathbb{P}[\nu(u,v) > t] \le \frac{Cl}{\sqrt{t}}.$$

Proof. Put $e_1 := (0,1)$. Since $\{\nu(u,v) > t\} \subset \cup_{i=1}^{l} \{\nu\big((i-1)e_1, ie_1\big) > t\}$ we have that

$$\mathbb{P}[\nu(u,v) > t] \le \sum_{i=1}^{l} \mathbb{P}\Big[\nu\big((i-1)e_1, ie_1\big) > t\Big] \le \frac{Cl}{\sqrt{t}}. \tag{4.1}$$

□

Define

$$Y_0^m := m \text{ and } Y_n^m := X_{\tau_n(u_m)}(u_m)(1) - X_{\tau_n(u_0)}(u_0)(1) \text{ for } n \ge 1.$$

and put $\nu^Y := \inf\{n \ge 1; Y_n^1 = 0\}$. The process Y^m represents the distance between the paths $\pi^{(0,0)}$ and $\pi^{(0,m)}$ on the renewal times for the pair $(\pi^{(0,0)}, \pi^{(0,m)})$. Now the proof of Proposition 4.1 follows directly from the next Lemma.

Lemma 4.1. *There exists positive constants* C_1 *and* C_2 *such that*

$$\mathbb{P}[\nu^Y > k] \le \frac{C_1}{\sqrt{k}} \tag{4.2}$$

and

$$\mathbb{P}[T_{\nu^Y} > k] \leq \frac{C_2}{\sqrt{k}} \text{ for all } k \geq 1. \tag{4.3}$$

for every $k \geq 1$, where $(T_n)_{n\geq 1}$ are the renewall times defined in the statement of Corollary 3.1 for the points $(0,0)$ *and* $(0,1)$.

To prove Lemma 4.1, we use a Skorohood's Representation of Y^m following ideas presented in [CFD09] and [CV14]. By the Skorohood's Representation theorem (see the Theorem 8.7.1 in [Dur13]) there exist a Brownian motion $(B(s))_{s\geq 0}$ starting in m and stopping times $(S_i)_{i\geq 0}$ such that

$$B(S_i) \stackrel{d}{=} Y_i, \text{ for } i \geq 0$$

and $(S_i)_{i\geq 0}$ has the following representation:

$$S_0 := 0, S_i := \inf\Big\{s \geq S_{i-1}; B(s) - B(S_{i-1}) \notin \big(U_i(B(S_{i-1})), V_i(B(S_{i-1}))\big)\Big\}$$

where $\big\{(U_i(m), V_i(m)); m \in \mathbb{Z}, i \geq 1\big\}$ is a family of independent random vectors taking values in $\big((\mathbb{Z}_- - \{0\}) \times \mathbb{N}\big) \cup \big\{(0,0)\big\}$.

Before we go to the proof Lemma 4.1, we still need the next two technical lemmas whose proofs will be postponed to Appendix B.

Lemma 4.2. *Let be $u_0 := (0,0)$ and for $m \in \mathbb{N}$ take $u_m := (m,0)$. Consider the sequences $\{\tau_n(u_0)\}_{n\geq 1}$ and $\{\tau_n(u_m)\}_{n\geq 1}$ as the renewals times introduced in Corollary 3.1 for the pair (u_0,u_m). Let us take $\nu^m_{(-\infty,0]}$ as the first time that $\{Y^m_n\}_{n\geq 1}$ is non-positive; i.e $\nu^m_{(-\infty,0]} := \inf\{n \geq 1; Y^m_n \leq 0\}$. Then*

(i) *For all $m \in \mathbb{N}$ we have $\mathbb{P}\big[\nu^m_{(-\infty,0]} < \infty\big] = 1$.*
(ii) $\inf_{m\geq 1} \mathbb{P}\big[Y^m_{\nu^m_{(-\infty,0]}} = 0\big] > 0$.
(iii) *Let us define the sequence $(a_l)_{l\geq 1}$ as*

$$a_1 := \inf\{n \geq 1; Y^1_n \leq 0\} \text{ and for } l \geq 2,$$

and

$$a_l := \begin{cases} \inf\{n \geq a_{l-1}; Y^1_n \geq 0\}; & \text{if } l \text{ is even} \\ \inf\{n \geq a_{l-1}; Y^1_n \leq 0\}; & \text{if } l \text{ is odd} \end{cases}.$$

Then there exists a constant $c_5 < 1$ such that $\mathbb{P}[Y^1_{a_j} \neq 0, \text{ for } j = 1,\dots,k] \leq c_5^k$ for all $k \geq 1$.

Lemma 4.3. *Consider the sequence $(a_l)_{l\geq 1}$ as in the previous lemma and $(S_n)_{n\geq 1}$ obtained from the Skorohood's Representation of Y^1. Then there exist a standard Brownian motion $(\mathbb{B}(s))_{s\geq 0}$ and random variables $(R_i)_{i\geq 1}$ and $\widetilde{R}_0$ independent of $(\mathbb{B}(s))_{s\geq 0}$ such that*

(i) $R_i | \{R_i \neq 0\} \stackrel{d}{=} \widetilde{R}_0$ *for all $i \geq 1$.*
(ii) *S_{a_l} is stochastically dominated by J_l which is defined as $J_0 = 0$,*

$$J_1 := \inf\{s \geq 0; \mathbb{B}(s) - \mathbb{B}(0) = -(R_1 + \widetilde{R}_0)\},$$

and

$$J_l := \inf\{s \geq J_{l-1}; \mathbb{B}(s) - \mathbb{B}(J_{l-1}) = (-1)^l (R_l + R_{l-1})\}, l \geq 2.$$

(iii) *Given that $\mathbb{B}(0) = \widetilde{R}_0$, $Y_{a_l} \neq 0$ implies $\mathbb{B}(J_l) \neq 0$.*

Proof of Lemma 4.1. Let us suppose that (4.2) is true and use it to prove (4.3) with the same idea used in [RSS16]. Recall from Corollary 3.1 that T_1 has finite moments and define the constant $L := 1/2\mathbb{E}[T_1]$ and take $k \in \mathbb{N}$ then

$$\mathbb{P}[T_{\nu^Y} > k] \leq \mathbb{P}[T_{\nu^Y} > k, \nu^Y \leq Lk] + \mathbb{P}[\nu^Y > Lk] \leq \mathbb{P}[T_{\lfloor Lk \rfloor} > k] + \mathbb{P}[\nu^Y > Lk].$$

By (4.2), it is enough to proof that $\mathbb{P}[T_{\lfloor Lk \rfloor} > k] \leq \frac{C_3}{\sqrt{k}}$ for some constant C_3. Then

$$\begin{aligned}\mathbb{P}\big[T_{\lfloor Lk \rfloor} > k\big] &= \mathbb{P}\Big[\sum_{i=1}^{\lfloor Lk \rfloor}[T_i - T_{i-1}] > k\Big] = \mathbb{P}\Big[\sum_{i=1}^{\lfloor Lk \rfloor}[T_i - T_{i-1}] - \lfloor Lk \rfloor \mathbb{E}[T_1] > k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big] \\ &\leq \frac{\mathbb{V}ar\Big[\sum_{i=1}^{\lfloor Lk \rfloor}(T_i - T_{i-1})\Big]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} = \frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2}.\end{aligned}$$

Note that

$$\sqrt{k}\frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \to 0 \text{ as } k \to 0.$$

Then there exist M such that

$$\frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \leq \frac{1}{\sqrt{k}}$$

for all $k \geq M$. Then we can find a sufficiently large constant $C_3 > 0$ such that

$$\frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \leq \frac{C_3}{\sqrt{k}}$$

for all $k \geq 1$.

Now we prove (4.3). Here we simply write $Y = Y^1$. Recall the definition of $(B(s))_{s\geq 0}$ and $(S_i)_{i\geq 0}$ above for the case $m = 1$. For $\delta > 0$ to be fixed later and every $k \in \mathbb{N}$ we have that,

$$\mathbb{P}[\nu^Y > k] = \mathbb{P}[S_k \leq \delta k, \nu^Y > k] + \mathbb{P}[S_k > \delta k, \nu^Y > k]. \tag{4.4}$$

Let us get an upper bound to $\mathbb{P}[S_k \leq \delta k, \nu^Y > k]$. From the Skorohod representation

$$S_k = \sum_{i=1}^{k}\big(S_i - S_{i-1}\big) = \sum_{i=1}^{k} Q_i(Y_{i-1}),$$

where $\big\{Q_i(m); i \geq 1, m \in \mathbb{Z}\big\}$ are independent random variables and $Q_i(m)$ is independent of $(Y_1, ..., Y_{i-1})$ for all $i \in \mathbb{N}, m \in \mathbb{Z}$. Note that on $\{\nu^Y > k\}$ we have that $Y_i \neq 0$ for $i \in \{1, \dots, k\}$.

Let us start considering the first term in (4.4). Fix $\lambda > 0$, then

$$\mathbb{P}[S_k \leq \delta k, \nu^Y > k] = \mathbb{P}[e^{-\lambda S_k} \geq e^{-\lambda \delta k}, \nu^Y > k] \leq e^{\lambda \delta k}\mathbb{E}[e^{-\lambda S_k} \quad {}_{\{\nu^Y > k\}}]$$

Claim 4.1.

$$\mathbb{E}[e^{-\lambda S_k} \quad {}_{\{\nu^Y > k\}}] \leq \Big(\sup_{m \in \mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q(m)}]\Big)^k,$$

where, for each m, $Q(m)$ is a random variable with the same distribution of $Q_1(m)$.

Proof of Claim 4.1. The proof is essentially the same given in the Theorem 4 in [CFD09]. We include it here for the sake of completeness, taking $\mathcal{F}_k := \sigma(Y_1, \dots, Y_k)$ we have that

$$\begin{aligned}
\mathbb{E}[e^{-\lambda S_k} \;{}_{\{\nu^Y > k\}}] &= \mathbb{E}\Big[\mathbb{E}[e^{-\lambda S_k} \;{}_{\{\nu^Y > k\}} | \mathcal{F}_{k-1}]\Big] \\
&\leq \mathbb{E}\Big[e^{-\lambda S_{k-1}} \mathbb{E}[e^{-\lambda Q_k(Y_{k-1})} \;{}_{\{\nu^Y > k-1\}} \;{}_{\{Y_{k-1} \neq 0\}} | \mathcal{F}_{k-1}]\Big] \\
&= \mathbb{E}\Big[e^{-\lambda S_{k-1}} \;{}_{\{\nu^Y > k-1\}} \mathbb{E}[e^{-\lambda Q_k(Y_{k-1})} \;{}_{\{Y_{k-1} \neq 0\}} | \mathcal{F}_{k-1}]\Big]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[e^{-\lambda Q_k(Y_{k-1})} \;{}_{\{Y_{k-1} \neq 0\}} | \mathcal{F}_{k-1}] &= \sum_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{E}[e^{-\lambda Q_k(m)} \;{}_{\{Y_{k-1} = m\}} | \mathcal{F}_{k-1}] \\
&= \sum_{m \in \mathbb{Z} \setminus \{0\}} \;{}_{\{Y_{k-1} = m\}} \mathbb{E}[e^{-\lambda Q_k(m)} | \mathcal{F}_{k-1}] \\
&= \sum_{m \in \mathbb{Z} \setminus \{0\}} \;{}_{\{Y_{k-1} = m\}} \mathbb{E}[e^{-\lambda Q(m)}] \\
&\leq \sup_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{E}[e^{-\lambda Q(m)}].
\end{aligned}$$

So, applying the above argument recursively we obtain

$$\mathbb{E}[e^{-\lambda S_k} \;{}_{\{\nu^Y > k\}}] \leq \mathbb{E}[e^{-\lambda S_{k-1}} \;{}_{\{\nu^Y > k-1\}}] \Big(\sup_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{E}[e^{-\lambda Q(m)}] \Big) \leq \Big(\sup_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{E}[e^{-\lambda Q(m)}] \Big)^k.$$

□

Using Claim 4.1 we get that

$$\mathbb{P}[S_k \leq \delta k, \nu^Y > k] \leq \Big(e^{\lambda \delta} \sup_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{E}[e^{-\lambda Q(m)}] \Big)^k.$$

Let $Q_{-1,1}$ be the exit time of interval $(-1, 1)$ by a Standard Brownian motion. Using Claim 4.2 we have that

$$\begin{aligned}
\mathbb{E}\big[e^{-\lambda Q(m)}\big] &= \mathbb{E}\big[e^{-\lambda Q(m)} | (U(m), V(m)) \neq (0,0))\big] \mathbb{P}[(U(m), V(m)) \neq (0,0)] \\
&\quad + \mathbb{P}\big[(U(m), V(m)) = (0,0)\big] \\
&\leq \mathbb{E}\big[e^{-\lambda Q_{-1,1}}\big] \Big(1 - \mathbb{P}\big[(U(m), V(m)) = (0,0)\big] \Big) + \mathbb{P}\big[(U(m), V(m)) = (0,0)\big] \\
&= \mathbb{P}\big[(U(m), V(m)) = (0,0)\big](1 - c_2) + c_2 \,,
\end{aligned} \tag{4.5}$$

where $c_2 = \mathbb{E}\big[e^{-\lambda Q_{-1,1}}\big] < 1$. Here we need the following:

Claim 4.2. $0 < c_1 := \sup_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{P}\big[(U(m), V(m)) = (0,0)\big] < 1$.

Using Claim 4.2 and (4.5), we obtain that

$$\mathbb{E}\big[e^{-\lambda Q(m)}\big] \leq c_1(1 - c_2) + c_2 \,.$$

Now chose δ such that $c_3 := e^{\delta \lambda}\big[c_1(1 - c_2) + c_2\big] < 1$. Then

$$\mathbb{P}[S_k \leq \delta k, \nu^Y > k] \leq c_3^k \leq \frac{c_4}{\sqrt{k}}, \tag{4.6}$$

for some suitable $c_4 > 0$. This gives the bound we need on the first term of (4.4). Let just prove the previous claim before dealing with the second term in (4.4).

Proof of Claim 4.2. The proof uses the hypothesis that $P(W=1)>0$. However by a straightforward adaptation, one can see that this is not required for the Claim to remain valid.

We follow the idea used in [RSS16]. For all $m \in \mathbb{Z}\setminus\{0\}$ we have

$$\mathbb{P}[\big(U(m),V(m)\big)=(0,0)] \geq (p\mathbb{P}[W=1])^2,$$

see Figure 5.

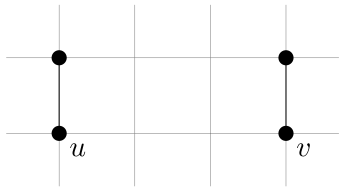

FIGURE 5. If $W_u=W_v=1$ and $u+e_2$ and $v+e^2$ are open, which occurs with probability $(p\mathbb{P}[W=1])^2$, then $\big(U(m),V(m)\big)=(0,0)$.

So $c_1 := \sup_{m\geq 1}\Big\{\mathbb{P}[\big(U(m),V(m)\big)=(0,0)]\Big\} \geq (p\mathbb{P}[W=1])^2>0$.

For the upper bound in the statement we have that

$$\mathbb{P}[\big(U(m),V(m)\big)\neq(0,0)] \geq (1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3,$$

see Figure 6.

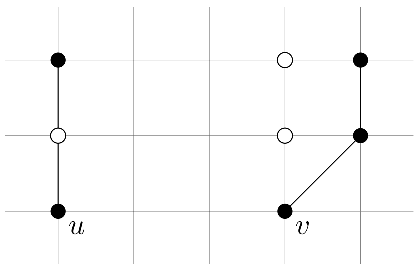

FIGURE 6. If $W_u=W_v=W_{v+e_1+e_2}=1$ and $u+2e_2$, $v+e_1+e_2$ and $v+e_1+2e_2$ are open and $u+e_2$, $v+e_2$ and $v+2e_2$ are closed, then with $p^3(1-p)^4(\mathbb{P}[W=1])^3$ if $|m|=1$ and $(1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3$ in case $|m|\neq 1$ we have the trajectory as represented in the picture where $\big(U(m),V(m)\big)=(0,0)$.

Hence $\inf_{m\in\mathbb{Z}\setminus\{0\}}\mathbb{P}[\big(U(m),V(m)\big)\neq(0,0)] \geq (1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3$. So,

$$c_1 \leq 1-(1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3<1.$$

$\square$

Now we consider the second term in (4.4). To deal with it we consider an approach similar to [CV14]. Take the sequence $(a_l)_{l\geq 0}$ as in the statement of Lemma 4.2. Note that

$$\mathbb{P}[\nu^Y>k, S_k>\delta k] \leq \sum_{l=1}^{k}\mathbb{P}[\nu^Y>k, S_k\geq \delta k, S_{a_{l-1}}<\delta k, S_{a_l}\geq \delta k]\,. \tag{4.7}$$

For now fix $l = 1, \dots k$, using Lemma 4.3 we get,

$$\{\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta\} \subseteq \{Y_{a_j} \neq 0, \text{ for } j = 1, \dots, l-1, S_{a_l} \geq k\delta\},$$

hence

$$\begin{aligned}\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] &\leq \mathbb{P}[Y_{a_j} \neq 0, \text{ for } j = 1, \dots, l-1, S_{a_l} \geq k\delta] \\ &\leq \mathbb{P}[Y_{a_j} \neq 0, \text{ for } j = 1, \dots l-1, J_l \geq k\delta] \\ &= \mathbb{P}[J_l \geq k\delta | Y_{a_j} \neq 0, \text{ for } j = 1, \dots l-1]\mathbb{P}[Y_{a_j} \neq 0, \text{ for } j = 1, \dots l-1]\end{aligned}$$

By the item $iii)$ in the Lemma 4.2 we get

$$\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \leq c_5^{l-1}\mathbb{P}[J_l \geq k\delta | Y_{a_j} \neq 0, \text{ for } j = 1, \dots l-1].$$

Now take $(\widetilde{R}_j)_{j\geq 1}$ i.i.d. random variables independent of $(B(s))_{s\geq 0}$ such that $\widetilde{R}_1 =^d R_0$ and define $\widetilde{J}_0 = 0$,

$$\widetilde{J}_j := \inf\{s \geq \widetilde{J}_{j-1}; B(s) - B(\widetilde{J}_{j-1}) = (-1)^l(\widetilde{R}_l + \widetilde{R}_{l-1})\}, j \geq 1.$$

We have that

$$\mathbb{P}[J_j \geq k\delta | Y_{a_j} \neq 0, \text{ for } j = 1, \dots l-1] = \mathbb{P}[\widetilde{J}_j \geq k\delta].$$

Taking $D_i := \widetilde{J}_i - \widetilde{J}_{i-1}$ for $i \geq 1$ and observing that $(D_i)_{i\geq 1}$ is an i.d. sequence we have that

$$\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \leq c_5^{l-1}\mathbb{P}[\widetilde{J}_l \geq k\delta] \leq c_5^{l-1} l\mathbb{P}[D_1 \geq \frac{k\delta}{l}].$$

Claim 4.3. *There exists constant $c_7 > 0$ such that for every $x > 0$ we have that*

$$\mathbb{P}[D_1 \geq x] \leq \frac{c_7}{\sqrt{x}}.$$

Proof of Claim 4.3. As in [CV14] take $\mu := \mathbb{E}[\widetilde{R}_0]$ and $\mathcal{J}_m := \inf\{t \geq 0; B(t) = m\}$. Then,

$$\begin{aligned}\mathbb{P}[D_1 \geq x] &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{P}[D_1 \geq x | \widetilde{R}_0 = k, \widetilde{R}_1 = j]\mathbb{P}[\widetilde{R}_0 = k, \widetilde{R}_1 = j] \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{P}[D_1 \geq x | \widetilde{R}_0 = k, \widetilde{R}_1 = j]\mathbb{P}[\widetilde{R}_0 = k]\mathbb{P}[\widetilde{R}_1 = j] \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{P}[\mathcal{J}_{k+j} \geq x]\mathbb{P}[\widetilde{R}_0 = k]\mathbb{P}[\widetilde{R}_1 = j] \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_x^{\infty}\frac{j+k}{\sqrt{2\pi y^3}}e^{-\frac{j+k}{2y}}dy\mathbb{P}[\widetilde{R}_0 = k]\mathbb{P}[\widetilde{R}_1 = j] \\ &\leq 2\mu^2\int_x^{\infty}\frac{1}{\sqrt{2\pi y^3}}e^{-\frac{1}{2y}}dy \\ &\leq \frac{c_7}{\sqrt{x}}.\end{aligned}$$

For some suitable constant c_7. □

Using Claim 4.3 we have some constant c_8 such that

$$\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \leq \frac{c_8 c_5^l l^{\frac{3}{2}}}{\sqrt{k}},$$

then

$$\mathbb{P}[\nu^Y > k, S_k > \delta k] \leq \sum_{l=1}^{k} \frac{c_8 c_5^l l^{\frac{3}{2}}}{\sqrt{k}} \leq \frac{c_8}{\sqrt{k}} \sum_{l=1}^{\infty} c_5^l l^{\frac{3}{2}} = \frac{c_9}{\sqrt{k}}. \tag{4.8}$$

Then we get that

$$\mathbb{P}[\nu^Y > k] \leq \frac{c_4 + c_9}{\sqrt{k}}.$$

□

5. The condition I.

In this section we will prove the condition I of the Theorem 2.2. First we need to obtain the constants γ and σ such that π_n^u as defined in (2.2) converges in distribution to a Brownian motion. If we have Corollary 3.1 which gave us the existence of the renewals times and a uniform bound on a moment of order higher than two for the displacement of paths on these renewal times, then the proof of the convergence of π_n^u is analogous to the one made in [RSS16]. Finally to get the condition I we will follow the ideas used in [CV14], the difference here is the need to work with the renewals times to make the coupling.

Proposition 5.1. *There exist positive constants γ and σ such that for any $u \in \mathbb{Z}^2$ the rescaled path π_n^u, as defined in* (2)*, converges in distribution to a Brownian motion starting in u as n goes to infinity.*

Proof. Without lost of generality we can assume that $u = (0,0)$. To make easier the notation we will omit u from the notation, i.e. we will write X_n instead of $X_n(u)$, $\pi(t)$ instead of $\pi^u(t)$ and so on. Taking $T_0 := 0, \tau_0 := 0$ and $(T_n)_{n\geq 1}$, $(\tau_n)_{n\geq 1}$ as defined in Corollary 3.1 for one point. Let $\widetilde{\pi}$ be the linear interpolation of the values of $(\pi(t))_{t\geq 0}$ on the renewals times,

$$\widetilde{\pi}(t) := \pi(T_n) + \frac{t - T_n}{T_{n+1} - T_n}\Big[\pi(T_{n+1}) - \pi(T_n)\Big], \text{ for } T_n \leq t \leq T_{n+1}.$$

Let us define the following random variables

$$Y_i := \pi(T_i) - \pi(T_{i-1}) \text{ for } i \geq 1.$$

$$S_0 := 0, S_n := \sum_{i=1}^{n} Y_i \text{ for } n \geq 1.$$

Let $\sigma^2 = \mathbb{V}ar(Y_1)$, then by Donsker's invariance principle we have that $(\widehat{\pi}_n(t))_{t\geq 0}$, defined as

$$\widehat{\pi}_n(0) := 0,\, \widehat{\pi}_n(t) := \frac{1}{n\sigma}\Big[(n^2 t - \lfloor n^2 t\rfloor)Y_{\lfloor n^2 t\rfloor + 1} + S_{\lfloor n^2 t\rfloor}\Big] \text{ for } t > 0\,,$$

converges in distribution as $n \to \infty$ to a Standard Brownian motion $(B(t))_{t\geq 0}$.
Put

$$A(t) := j + \frac{t - T_j}{T_{j+1} - T_j} \text{ for } T_j \leq t < T_{j+1}, \text{ and } N(t) := \sup\{n \geq 1; T_n \leq t\} \text{ for } t > 0.$$

Note that $N(t) \leq A(t) \leq N(t) + 1$ for all $t > 0$. Since $T_n = \sum_{j=1}^{n}(T_j - T_{j-1})$, by Corollary 3.1, $(N(t))_{t\geq 0}$ is a renewal process. By the Renewal Theorem $\frac{N(t)}{t} \to \frac{1}{\mathbb{E}[T_1]}$ as $t \to \infty$ almost surely, hence for $\gamma = \mathbb{E}[T_1]$ we have that $\frac{A(n^2\gamma t)}{n^2} \to t$ almost surely. For $n \geq 1$ let us rescale $\widetilde{\pi}$ as

$$\widetilde{\pi}_n(t) := \frac{\widetilde{\pi}(\gamma n^2 t)}{n\sigma} \text{ for } t \geq 0.$$

Note that

$$\widetilde{\pi}_n(t) = \widehat{\pi}_n\Big(\frac{A(n^2\gamma t)}{n^2}\Big) \text{ for } t \geq 0.$$

We have that $(\widetilde{\pi}_n(t))_{t\geq 0}$ converges in distribution to a $(B(t))_{t\geq 0}$, we leave the details to the reader. To prove the convergence of $(\pi_n(t))_{t\geq 0}$ to $(B(t))_{t\geq 0}$ it is enough to show that for any $\epsilon > 0$ and $s > 0$, $\mathbb{P}\big[\sup_{0\leq t\leq s}|\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\big] \to 0$ as $n\to\infty$. Note that

$$\mathbb{P}\Big[\sup_{0\leq t\leq s}|\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\Big] = \mathbb{P}\Big[\sup_{0\leq t\leq sn^2\gamma}|\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big].$$

Moreover

$$\Big\{\sup_{0\leq t\leq sn^2\gamma}|\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\} \subseteq \bigcup_{j=0}^{N(sn^2\gamma)}\Big\{\sup_{T_j\leq t\leq T_{j+1}}|\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\}$$

and $N(sn^2\gamma) \leq \lfloor sn^2\gamma\rfloor$ so,

$$\Big\{\sup_{0\leq t\leq sn^2\gamma}|\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\} \subseteq \bigcup_{j=0}^{\lfloor sn^2\gamma\rfloor}\Big\{\sup_{T_j\leq t\leq T_{j+1}}|\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\}.$$

Since π and $\widetilde{\pi}$ coincide at the renewal times and their increments are stationary then

$$\mathbb{P}\Big[\sup_{0\leq t\leq s}|\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\Big] \leq (\lfloor sn^2\gamma\rfloor + 1)\mathbb{P}\Big[\sup_{0\leq t\leq T_1}\{|\pi(t) - \widetilde{\pi}(t)|\} > \epsilon n\sigma\Big].$$

Note that $\pi(T_1) = \widetilde{\pi}(T_1)$ and $\sup_{0\leq t\leq T_1}|\pi(t) - \pi(T_1)| \leq Z$, $\sup_{0\leq t\leq T_1}|\widetilde{\pi}(t) - \pi(T_1)| \leq Z$, where Z is defined in Proposition 3.1. Then

$$\mathbb{P}\Big[\sup_{0\leq t\leq s}|\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\Big] \leq (\lfloor sn^2\gamma\rfloor + 1)\mathbb{P}[2Z > \epsilon n\sigma] \leq \frac{2^3(\lfloor sn^2\gamma\rfloor + 1)\mathbb{E}[Z^3]}{\epsilon^3\sigma^3 n^3} \to 0 \text{ as } n\to\infty.$$

□

Proposition 5.2. *Let $\mathcal{X}_n$ be defined as in* (2.2) *where the constants γ and σ are taken as in Proposition 5.1. Then for any $y_1,\dots,y_m \in \mathbb{R}^2$ there exist paths $\theta_n^{y_1},\dots,\theta_n^{y_m}$ in $\mathcal{X}_n$, such that $(\theta_n^{y_1},\dots,\theta_n^{y_m})$ converges in distribution as $n\to\infty$ to coalescing Brownian motions starting in $y_1,\dots,y_m$.*

To prove Proposition 5.2 we will use a coupling argument. To build the coupling, we will need Proposition 5.3 below, which is a version of Proposition 3.1 that will be presented without proof because its proof follows the same lines as those of Proposition 3.1.

Proposition 5.3. *Let $\{U_v^1; v\in\mathbb{Z}^2\}$, $\{U_v^2; v\in\mathbb{Z}^2\}$, $\{W_v^1; v\in\mathbb{Z}^2\}$ and $\{W_v^2; v\in\mathbb{Z}^2\}$ be i.i.d. families independent of each other such that the U_v^j, $j=1,2$, are Uniform random variables in $[0,1]$ and of W_v^j, $j=1,2$, are identically distributed positives random variables on $\mathbb{N}$ with finite support. Consider the GRDF systems*

$$\mathcal{X}^1 := \{\pi^{1,v}, v\in\mathbb{Z}^2\} \text{ and } \mathcal{X}^2 := \{\pi^{2,v}; v\in\mathbb{Z}^2\}$$

built respectively using the random variables $\big\{\{U_v^1; v\in\mathbb{Z}^2\},\{W_v^1; v\in\mathbb{Z}^2\}\big\}$ and $\big\{\{U_v^2; v\in\mathbb{Z}^2\},\{W_v^2; v\in\mathbb{Z}^2\}\big\}$. Then for points $u_1^1,\dots,u_{m_1}^1$ and $u_1^2,\dots,u_{m_2}^2$ in $\mathbb{Z}^2$ at the same time level, i.e. with equal second component, there exist random variables T, Z and $\tau(u_i^j)$ for $j=1,2$, $1\leq i\leq m_j$, such that $T\leq Z$ and

(i) $\Delta^j_{\tau(u^j_i)}(u_i) = \emptyset$ *and* $X^1_{\tau(u^1_1)}(u^1_1)(2) = X^j_{\tau(u^j_i)}(u^j_i)(2)$ *for* $j = 1, 2$ *and* $i = 1, \ldots, m_j$. *Where for* $j = 1, 2$ *and* $v \in \mathbb{Z}^2$ *the sequence* $\{X^j_k(v)\}_{k\geq 0}$ *is as defined in (2) using the r.v.* $\{U^j_v; v \in \mathbb{Z}^2\}$, $\{W^j_v; v \in \mathbb{Z}^2\}$ *and* $\{\Delta^j_k(v)\}_{k\geq 0}$ *as defined in (3.1) for the sequence* $\{X^j_k(v)\}_{k\geq 0}$.

(ii) *Taking* $T := X^1_{\tau(u^1_1)}(u^1_1)(2)$ *we have that its distribution depends on* $m_1 + m_2$ *but not on* $u^j_1, \ldots, u^j_{m_j}$, $j = 1, 2$. *For all* $k \geq 1$ *we get* $\mathbb{E}\big[T^k\big] < \infty$. *Note that* $\pi^{j,u^j_i}(T) = X^j_{\tau(u^j_i)}(u^j_i)(1)$ *for* $j = 1, 2$, $1 \leq i \leq m_j$.

(iii) *For all* $j = 1, 2$ *and* $1 \leq i \leq m_j$ $i = 1, \ldots, m$ *we have that* $\sup_{0\leq t\leq T} |\pi^{j,u^j_i}(t) - u^j_i(1)| \leq Z$ *and its distribution depends on* $m_1 + m_2$ *but not on* $u^j_1, \ldots, u^j_{m_j}$ *for* $j = 1, 2$. *Also for all* $k \geq 1$ *we get* $\mathbb{E}\big[Z^k\big] < \infty$.

Proof of the Proposition 5.2. Here we use a non-straightforward adaptation of the idea applied in [CV14] to proof the condition I for the Drainage Network model. We will prove that for any $m \in \mathbb{N}$,

$$(\pi^{(0,0)}_n, \pi^{(n\sigma,0)}_n, \ldots, \pi^{(mn\sigma,0)}_n)$$

converges in distribution to a vector of coalescing Brownian motions starting in $(0, 0), \ldots, (m, 0)$ denoted here by $(B^{(0,0)}, \ldots, B^{(m,0)})$. The general case, where the paths do not start necessarily at the same time, could be proved using the same technique, so we will omit it. To simplify the notation we will write $\pi^k := \pi^{(k,0)}$, $k \in \mathbb{Z}$, and $B^x := B^{(x,0)}$ for $x \in \mathbb{R}$. Here for the rescaled paths we use the notation:

$$\pi^k_n = \frac{\pi^{k\lfloor n\sigma\rfloor}(tn^2\gamma)}{n\sigma}.$$

It is enough to fix an arbitrary $M > 0$, suppose that $(B^0, \ldots, B^m)$ and $(\pi^0_n, \pi^1_n, \ldots, \pi^m_n)$ are restricted to time interval $[0, M]$ and prove the convergence, i.e.,

$$\lim_{n\to\infty} (\pi^0_n(t), \pi^1_n(t), \ldots, \pi^m_n(t))_{0\leq t\leq M} \stackrel{d}{=} \big(B^0(t), \ldots, B^m(t)\big)_{0\leq t\leq M} \tag{5.1}$$

By Proposition 5.1 we have that

$$\lim_{n\to\infty} (\pi^0(t))_{0\leq t\leq M} \stackrel{d}{=} \big(B^0(t)\big)_{0\leq t\leq M}.$$

Now we are going to make an induction in m. Let us suppose that

$$\lim_{n\to\infty} (\pi^0_n(t), \pi^1_n(t), \ldots, \pi^{(m-1)}_n(t))_{0\leq t\leq M} \stackrel{d}{=} \Big(B^0(t), \ldots, B^{(m-1)}(t)\Big)_{0\leq t\leq M}.$$

Now the proof of (5.1) from the induction hypothesis will be based on coupling techniques. We will build a path $\overline{\pi}^m_n$ which is independent of $(\pi^0_n, \pi^1_n, \ldots, \pi^{(m-1)}_n)$ until coalescence with one of them, has the same distribution of π^m and such that, in a proper way, π_n and $\overline{\pi}_n$ are close to each other.

We start constructing paths $\widetilde{\pi}^0$, ... , $\widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}$ and $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ that coincide with $(\widetilde{\pi}^0, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor})$ until one of these paths moves a distance $n^{\frac{3}{4}}$ from its last position on the renewal times, we suggest the reader to see Figure 7 although some definitions are still missing. The construction follows by induction:

Step 1: Let $\{\widetilde{U}_v; v \in \mathbb{Z}^2\}$ and $\{\widehat{U}_v; v \in \mathbb{Z}^2\}$ be i.i.d. families of Uniform r.v. in $[0, 1]$; $\{\widetilde{W}_v; v \in \mathbb{Z}^2\}$ and $\{\widehat{W}_v; v \in \mathbb{Z}^2\}$ be i.i.d families of r.v. with the same distribution of $W_{(0,0)}$; independent of each other and of $\{U_v; v \in \mathbb{Z}^2\}$ and $\{W_v; v \in \mathbb{Z}^2\}$. Using them let us define the r.v. $\{\widetilde{U}^1_v; v \in \mathbb{Z}^2\}$, $\{\widehat{U}^1_v; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}^1_v; v \in \mathbb{Z}^2\}$ and $\{\widehat{W}^1_v; v \in \mathbb{Z}^2\}$ as follows:

$$\widehat{U}^1_v := \begin{cases} U_v; \text{ if } |v(1) - mn\sigma| \leq n^{\frac{3}{4}} \text{ and } 0 < v(2) \leq n^{\frac{3}{4}}; \\ \widehat{U}_v; \text{ otherwise,} \end{cases}$$

$$\widehat{W}_v^1 := \begin{cases} W_v; \text{ if } |v(1) - mn\sigma| \le n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widehat{W}_v; \text{ otherwise,} \end{cases}$$

$$\widetilde{W}_v^1 := \begin{cases} W_v; \text{ if } v(1) \le (m-1)n\sigma + n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widetilde{W}_v; \text{ otherwise,} \end{cases}$$

and

$$\widetilde{U}_v^1 := \begin{cases} U_v; \text{ if } v(1) \le (m-1)n\sigma + n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widetilde{U}_v; \text{ otherwise.} \end{cases}$$

Use the families $\{\widetilde{U}_v^1; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}_v^1; v \in \mathbb{Z}^2\}$ to construct a path $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ of the GRDF (not rescaled) starting in $m\lfloor n\sigma\rfloor$ at time zero. Also use $\{\widetilde{U}_v^1; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}_v^1; v \in \mathbb{Z}^2\}$ to construct paths $\{\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}\}$ of the GRDF (not rescaled) starting respectively in $0, \lfloor n\sigma\rfloor, \dots, (m-1)\lfloor n\sigma\rfloor$ at time zero. Let T_1 and Z_1 be the random variables associated to $\{\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor}\}$ by Proposition 5.3. Note that on the event $\{Z_1 \le n^{\frac{3}{4}}\}$ the vector paths $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor})$ coincide with $(\pi^0, \dots, \pi^{m\lfloor n\sigma\rfloor})$ up to time $T_1 \le Z_1$. This ends Step 1.

Step 2: See that nothing above T_1 is known, so we can use other random variables to define the paths after this time. So from time T_1, we define new independent iid families $\{\widetilde{U}_v^2; v \in \mathbb{Z}^2\}$, $\{\widehat{U}_v^2; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}_v^2; v \in \mathbb{Z}^2\}$ and $\{\widehat{W}_v^2; v \in \mathbb{Z}^2\}$ as follows:

$$\widehat{U}_v^2 := \begin{cases} U_v; \text{ if } |v(1) - \widehat{\pi}^{1,m\lfloor n\sigma\rfloor}(T_1)| \le n^{\frac{3}{4}} \text{ and } T_1 < v(2) \le T_1 + n^{\frac{3}{4}}; \\ \widehat{U}_v; \text{ otherwise,} \end{cases}$$

$$\widehat{W}_v^2 := \begin{cases} W_v; \text{ if } |v(1) - \widehat{\pi}^{1,m\lfloor n\sigma\rfloor}(T_1)| \le n^{\frac{3}{4}} \text{ and } T_1 < v(2) \le T_1 + n^{\frac{3}{4}}; \\ \widehat{W}_v; \text{ otherwise,} \end{cases}$$

$$\widetilde{W}_v^2 := \begin{cases} W_v; \text{ if } v(1) \le \max_{0\le j\le m-1} \widetilde{\pi}^{1,j\lfloor n\sigma\rfloor}(T_1) + n^{\frac{3}{4}} \text{ and } T_1 < v(2) \le T_1 + n^{\frac{3}{4}}; \\ \widetilde{W}_v; \text{ otherwise,} \end{cases}$$

and

$$\widetilde{U}_v^2 := \begin{cases} U_v; \text{ if } v(1) \le \max_{0\le j\le m-1} \widetilde{\pi}^{1,j\lfloor n\sigma\rfloor}(T_1) + n^{\frac{3}{4}} \text{ and } T_1 < v(2) \le T_1 + n^{\frac{3}{4}}; \\ \widetilde{U}_v; \text{ otherwise.} \end{cases}$$

Now consider $\widehat{\pi}^{2,m\lfloor n\sigma\rfloor}$ as the GRDF path starting in $\widehat{\pi}^{m\lfloor n\sigma\rfloor}(T_1)$ at time T_1 using the environment $\{\widehat{U}_v^2; v \in \mathbb{Z}^2\}$, $\{\widehat{W}_v^2; v \in \mathbb{Z}^2\}$, and $\widetilde{\pi}^{2,0}, \widetilde{\pi}^{2,\lfloor n\sigma\rfloor}, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}$ starting respectively in $\widetilde{\pi}^0(T_1), \widetilde{\pi}^{\lfloor n\sigma\rfloor}(T_1), \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}(T_1)$ and using the environment $\{\widetilde{U}_v^2, v \in \mathbb{Z}^2\}$,$\{\widetilde{W}_v^2; v \in \mathbb{Z}^2\}$. Again we have random variables T_2 and Z_2 for these paths as in Proposition 5.3 and on the event $\{\max(Z_1, Z_2) \le n^{\frac{3}{4}}\}$ the vector $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor})$ coincide with $(\pi^0, \dots, \pi^{m\lfloor n\sigma\rfloor})$ up to time $T_2 \le Z_1 + Z_2$. Redefine, if necessary, $(\widetilde{\pi}^0, \widetilde{\pi}^{\lfloor n\sigma\rfloor}, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor})$ as $(\widetilde{\pi}^{2,0}, \widetilde{\pi}^{2,\lfloor n\sigma\rfloor}, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor})$ on time interval $T_1 < t \le T_2$. This ends Step 2.

We continue step by step replicating recursively Step k from Step k-1. We get $(T_k)_{k\ge1}$, $(Z_k)_{k\ge1}$ and $\{\widetilde{\pi}^{k,0}, \widetilde{\pi}^{k,\lfloor n\sigma\rfloor}, \dots, \widetilde{\pi}^{k,(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{k,m\lfloor n\sigma\rfloor}\}$ for $k \ge 1$ such that on the event $\{\max(Z_1, ..., Z_k) \le n^{\frac{3}{4}}\}$ the vector $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor})$ coincide with $(\pi^0, \dots, \pi^{m\lfloor n\sigma\rfloor})$ up to time $T_k \le \sum_{j=1}^k Z_j$.

Now let us define a version $\overline{\pi}^{m\lfloor n\sigma\rfloor}$ of $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ such that it is independent of $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor})$ and coincide with $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ until this last path gets to distance $2n^{3/4}$ of $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor})$. Consider the following stopping time

$$\nu := \inf\Big\{k \ge 1; \max_{0\le j\le m-1} |\widehat{\pi}^{m\lfloor n\sigma\rfloor}(T_k) - \widetilde{\pi}^{j\lfloor n\sigma\rfloor}(T_k)| \le 2n^{\frac{3}{4}}\Big\}.$$

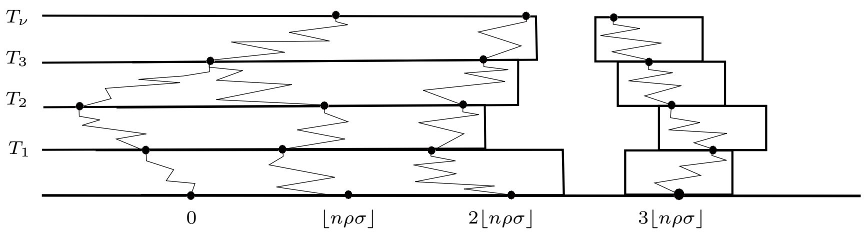

FIGURE 7. Here m=4 and we consider the GRDF paths π^0, $\pi^{\lfloor n\sigma\rfloor}$, $\pi^{2\lfloor n\sigma\rfloor}$. In the picture $\pi^{3\lfloor n\sigma\rfloor}$ remains at distance $n^{\frac{3}{4}}$ of its position on the previous renewal time. Moreover none of π^0, $\pi^{\lfloor n\sigma\rfloor}$, $\pi^{2\lfloor n\sigma\rfloor}$ go beyond $n^{\frac{3}{4}}$ to the right of their rightmost position at the previous renewal time. In such scenario, $\nu = 4$ and before time T_4 we have that $(\pi^0, \pi^{\lfloor n\sigma\rfloor}, \pi^{2\lfloor n\sigma\rfloor}, \pi^{3\lfloor n\sigma\rfloor})$ coincide with $(\widetilde{\pi}^0, \widetilde{\pi}^{\lfloor n\sigma\rfloor}, \widetilde{\pi}^{2\lfloor n\sigma\rfloor}, \widetilde{\pi}^{3\lfloor n\sigma\rfloor})$.

Define $\overline{\pi}^{m\lfloor n\sigma\rfloor}(t) = \widehat{\pi}^{m\lfloor n\sigma\rfloor}(t)$ for $0 \le t \le T_\nu$, see Figure 7. From time T_ν we have that $\overline{\pi}^{m\lfloor n\sigma\rfloor}(t)$ evolves only through the environment $(\{\widehat{U}_v; v \in \mathbb{Z}^2\}, \{\widehat{W}_v; v \in \mathbb{Z}^2\})$ as the path starting in $\widehat{\pi}^{m\lfloor n\sigma\rfloor}(T_\nu)$ at time T_ν before coalescence with some $\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}$. Let

$$\widetilde{\pi}_n^j(t) := \frac{\widetilde{\pi}^{j\lfloor n\sigma\rfloor}(tn^2\gamma)}{n\sigma} \text{ for } j = 0, \dots, m-1$$

and

$$\widehat{\pi}_n^m(t) := \frac{\widehat{\pi}^{m\lfloor n\sigma\rfloor}(t)}{n\sigma}, \overline{\pi}_n^m(t) := \frac{\overline{\pi}^{m\lfloor n\sigma\rfloor}(t)}{n\sigma}$$

the rescaled versions of the constructed paths.

Remark 5.1. *We point out that as a direct consequence of the definitions the following properties are satisfied:*

(i) *Before coalescence, the path $\overline{\pi}_n^m$ is independent of $\widetilde{\pi}_n^0, \dots, \widetilde{\pi}_n^{(m-1)}$.*
(ii) *For $s \le M$, on the event*
$$\mathcal{A}_{n,s} := \{T_\nu > n^2\gamma s\},$$
we have that $\widehat{\pi}_n^m(t) = \overline{\pi}_n^m(t)$ for every $0 \le t \le s$.
(iii) *From the induction hypothesis, item (i) and Proposition 5.1 we get*
$$\lim_{n\to\infty} \left(\widetilde{\pi}_n^0, \dots, \widetilde{\pi}_n^{(m-1)}, \overline{\pi}_n^m\right) \stackrel{d}{=} \left(B^0, \dots, B^m\right).$$
(iv) *On the event*
$$\mathcal{B}_{n,M} := \cap_{k=1}^{\lfloor Mn^2\gamma\rfloor+1} \{Z_k \le n^{\frac{3}{4}}\}$$
the vector of paths $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)}, \widehat{\pi}^m)$ coincide with $(\pi^0, \dots, \pi^m)$ up to a time greater than $Mn^2\gamma$.
(v) *Also on $\mathcal{B}_{n,M}$, if $|\widehat{\pi}^{m\lfloor n\sigma\rfloor}(t) - \widetilde{\pi}^{j\lfloor n\sigma\rfloor}(t)| \le 2n^{\frac{3}{4}}$ for some $0 \le j \le m-1$ and $t > 0$ then either there exists some k such that $T_k < t$, $|\widehat{\pi}^{m\lfloor n\sigma\rfloor}(T_k) - \widetilde{\pi}^{j\lfloor n\sigma\rfloor}(T_k)| \le 2n^{\frac{3}{4}}$ and $\nu \le k$ or $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ and $\widetilde{\pi}^{j\lfloor n\sigma\rfloor}$ cannot coalesce or cross each other before $\min\{T_k : T_k > t\}$.*

Claim 5.1. *For the event $\mathcal{B}_{n,M}$ as in Remark 5.1 we have that $\lim_{n\to\infty} \mathbb{P}\left[\mathcal{B}_{n,M}^c\right] = 0$.*

Proof. Note that

$$\mathbb{P}\left[\mathcal{B}_{n,M}^c\right] \le (Mn^2\gamma + 1)\mathbb{P}\left[Z_1 > n^{\frac{3}{4}}\right] \le \frac{(Mn^2\gamma + 1)\mathbb{E}[Z_1^4]}{n^3}$$

which goes to zero as n goes to infinity. ☐

Now let $H : C^{k+1}[0,M] \to \mathbb{R}$ be an uniformly continuous function. We need to prove that

$$\lim_{n\to\infty} \mathbb{E}\big[H\big(\pi_n^0,\dots,\pi_n^m\big)\big] = \mathbb{E}\big[H\big(B^0,\dots,B^m\big)\big].$$

By Remark 5.1 and Claim 5.1 we have that

$$\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^m\big) - H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)\big|\Big] \le 2||H||_\infty \mathbb{P}\big[\mathcal{B}_{n,M}^c\big] \to 0 \text{ as n goes to infinity.}$$

And also by Remark 5.1 and the induction hypothesis we have that

$$\mathbb{E}\Big[H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\overline{\pi}_n^m\big)\Big] \to \mathbb{E}\Big[H\big(B^0,\dots,B^m\big)\Big].$$

By triangular inequality

$$\begin{aligned}
&\Big|\mathbb{E}\big[H\big(\pi_n^0,\dots,\pi_n^m\big)\big] - \mathbb{E}\big[H\big(B^0,\dots,B^m\big)\big]\Big| \\
&\le \mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^m\big) - H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)\big|\Big] \\
&+ \mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big) - H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\overline{\pi}_n^m\big)\big|\Big] \\
&+ \Big|\mathbb{E}\Big[H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\overline{\pi}_n^m\big) - H\big(B^0,\dots,B^m\big)\Big]\Big|,
\end{aligned}$$

then it is enough to prove that

$$\lim_{n\to\infty} \mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big) - H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\overline{\pi}_n^m\big)\big|\Big] = 0\,.$$

Note that

$$\begin{aligned}
&\mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big) - H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\overline{\pi}_n^m\big)\big|\Big] \\
&= \mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big) - H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\overline{\pi}_n^m\big)\big| \;_{\mathcal{A}_{n,M}^c}\Big] \\
&\le \mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big) - H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\overline{\pi}_n^m\big)\big| \;_{\mathcal{A}_{n,M}^c}\;_{\mathcal{B}_{n,M}}\Big] + 2||H||_\infty \mathbb{P}\big[\mathcal{B}_{n,M}^c\big].
\end{aligned}$$

Again, by Claim 5.1 we just have to prove that

$$\lim_{n\to\infty} \mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big) - H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\overline{\pi}_n^m\big)\big| \;_{\mathcal{A}_{n,M}^c}\;_{\mathcal{B}_{n,M}}\Big] = 0\,.$$

Before we are able to obtain the above convergence, we need to define some stopping times. For $j = \{0,\dots,m-1\}$ consider

$$\nu_j := \inf\{k \ge 1 : |\pi^{j\lfloor n\sigma\rfloor}(T_k) - \pi^{m\lfloor n\sigma\rfloor}(T_k)| \le 2n^{\frac{3}{4}}\}\,,$$

where the definition is based on (v) in Remark 5.1 from where we see that on $\mathcal{B}_{n,M}$ we only need to consider approximation between paths on the renewal times. Then

$$\begin{aligned}
&\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big) - H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\overline{\pi}_n^m\big)\big| \;_{\mathcal{A}_{n,M}^c}\;_{\mathcal{B}_{n,M}}\Big] \\
&\le \sum_{j=0}^{m-1} \mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big) - H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\overline{\pi}_n^m\big)\big| \;_{\mathcal{A}_{n,M}^c}\;_{\mathcal{B}_{n,M}}\;_{\{\nu=\nu_j\}}\Big].
\end{aligned}$$

Given $\epsilon > 0$, since H is uniformly continuous, there exists $\delta_\epsilon > 0$ such that: if $\|f-g\|_\infty \le \delta_\epsilon$ for $f,g \in C^{k=1}[0,M]$, then $\big|H(f) - H(g)\big| \le \epsilon$. So, if

$$\sup_{0\le t\le M} |\pi_n^m(t) - \overline{\pi}_n^m(t)| \le \delta_\epsilon$$

we get

$$\Big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\overline{\pi}_n^m\big)\Big|\le\epsilon.$$

To simplify the notation let us denote $D_{n,j}:=\mathcal{A}_{n,M}^c\cap\mathcal{B}_{n,M}\cap\{\nu=\nu_j\}$. For $j=0,\dots,m-1$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\overline{\pi}_n^m\big)\big|\ {}_{D_{n,j}}\Big]\\
&\le\epsilon+2||H||_\infty\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le M}|\pi_n^m(t)-\overline{\pi}_n^m(t)|>\delta_\epsilon\}\Big]\\
&=\epsilon+2||H||_\infty\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\overline{\pi}^{m\lfloor n\sigma\rfloor}(t)|>n\sigma\delta_\epsilon\}\Big].
\end{aligned}$$

For $j=0,\dots,m-1$ let us define

$$\tau^j:=\inf\{t>0:\pi^{j\lfloor n\sigma\rfloor}(s)=\pi^{m\lfloor n\sigma\rfloor}(s),\ \forall s\ge t\}$$

and

$$\overline{\tau}^j:=\inf\{t>0:\pi^{j\lfloor n\sigma\rfloor}(s)=\overline{\pi}^{m\lfloor n\sigma\rfloor}(s),\ \forall s\ge t\}.$$

Fix some $\beta\in(\frac{3}{2},2)$. Then for $j=0,\dots,m-1$ and n large enough

$$\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\overline{\pi}^{m\lfloor n\sigma\rfloor}(t)|>n\sigma\delta_\epsilon\}\Big]$$

is bounded above by

$$\begin{aligned}
&\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\overline{\pi}^{m\lfloor n\sigma\rfloor}(t)|>n\sigma\delta_\epsilon\}\cap\{\tau^j,\overline{\tau}^j\in[T_\nu,T_\nu+n^\beta\gamma]\}\Big]\\
&+\mathbb{P}\Big[D_{n,j}\cap\{\tau^j>T_\nu+n^\beta\gamma]\}\Big]+\mathbb{P}\Big[D_{n,j}\cap\{\overline{\tau}^j>T_\nu+n^\beta\gamma]\}\Big].
\end{aligned}\tag{5.2}$$

The first probability in (5.2) is equal to

$$\mathbb{P}\Big[D_{n,j}\cap\{\sup_{T_{\nu_j}\le t\le Mn^2\gamma\wedge(T_{\nu_j}+n^\beta\gamma)}|\pi_n^m(t)-\overline{\pi}_n^m(t)|>n\sigma\delta_\epsilon\}\cap\{\tau^j,\overline{\tau}^j\in[T_{\nu_j},T_{\nu_j}+n^\beta\gamma]\}\Big]$$

which is bounded above by

$$\begin{aligned}
&\mathbb{P}\Big[D_{n,j}\cap\{\sup_{T_{\nu_j}\le t\le Mn^2\gamma\wedge(T_{\nu_j}+n^\beta\gamma)}|\pi_n^m(t)-\pi_n^m(T_{\nu_j})|>\frac{n\sigma\delta_\epsilon}{2}\}\cap\{\tau^j,\overline{\tau}^j\in[T_{\nu_j},T_{\nu_j}+n^\beta\gamma]\}\Big]\\
&+\mathbb{P}\Big[D_{n,j}\cap\{\sup_{T_{\nu_j}\le t\le Mn^2\gamma\wedge(T_{\nu_j}+n^\beta\gamma)}|\overline{\pi}_n^m(t)-\overline{\pi}_n^m(T_{\nu_j})|>\frac{n\sigma\delta_\epsilon}{2}\}\cap\{\tau^j,\overline{\tau}^j\in[T_{\nu_j},T_{\nu_j}+n^\beta\gamma]\}\Big]\\
&\le\mathbb{P}\Big[\sup_{0\le t\le n^\beta\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\pi^{m\lfloor n\sigma\rfloor}(0)|>\frac{n\sigma\delta_\epsilon}{2}\Big]+\mathbb{P}\Big[\sup_{0\le t\le n^\beta\gamma}|\overline{\pi}^{m\lfloor n\sigma\rfloor}(t)-\overline{\pi}^{m\lfloor n\sigma\rfloor}(0)|>\frac{n\sigma\delta_\epsilon}{2}\Big],
\end{aligned}$$

where for the inequality we have used the Markov property on the renewal times. Both terms in the right hand side of the previous inequality are bounded above by

$$\mathbb{P}\Big[\sup_{0\le t\le n^\beta\gamma}\frac{|\pi^0(t)-\pi^0(0)|}{\sigma n^{\frac{\beta}{2}}}>\frac{n^{1-\frac{\beta}{2}}\delta_\epsilon}{2}\Big],$$

which, by the choice of $\beta<2$ and the invariance principle proved in Proposition 5.1, converges to zero as $n\to\infty$. Thus the first probability in (5.2) converges to zero as $n\to\infty$.

Now it remains to deal with the second and third terms in (5.2). Since

$$|\pi^{j\lfloor n\sigma\rfloor}(T_{v_j})-\pi^{m\lfloor n\sigma\rfloor}(T_{v_j})|\le 2n^{\frac{3}{4}}$$

on $D_{n,j}$, then by Corollary 4.1 there is some constant C such that

$$\mathbb{P}\Big[D_{n,j} \cap \{\tau^j > T_{\nu_j} + n^\beta \gamma\}\Big] \leq \frac{2Cn^{\frac{3}{4}}}{n^{\frac{\beta}{2}}}$$

which converges to zero as $n \to \infty$ by the choice of $\beta > 3/2$. Even though $\pi^{j\lfloor n\sigma \rfloor}$ and $\overline{\pi}^{j\lfloor n\sigma \rfloor}$ are independent from time T_{ν_j} until coalescence, we can prove the result stated in Corollary 4.1 for these paths, following the same lines of the proof of that corollary. Thus we get a constant $\overline{C}$ such that

$$\mathbb{P}\Big[D_{n,j} \cap \{\overline{\tau}^j > T_{\nu_j} + n^\beta \gamma\}\Big] \leq \frac{2\overline{C}n^{\frac{3}{4}}}{n^{\frac{\beta}{2}}} \,,$$

which as before converges to zero as $n \to \infty$.

Hence

$$\lim_{n\to\infty} \mathbb{P}\Big[\mathcal{A}_{n,M}^c \cap \mathcal{B}_{n,M} \cap \{\nu = \nu_j\} \cap \{\sup_{0 \leq t \leq Mn^2\gamma} |\pi_n^m(t) - \overline{\pi}_n^m(t)| > n\sigma\delta_\epsilon\}\Big] = 0$$

which finishes the proof. □

6. The condition B

We prove condition B of Theorem 2.2 at the end of this section. Before we prove it we need to introduce some definitions and stablish some preliminary results.

Take $K \in \mathbb{N}$ such that $\mathbb{P}[W_{(0,0)} \leq K] = 1$, recall that we are supposing that the distribution of $W_{(0,0)}$ has finite support. For $u \in \mathbb{Z}^2$ define

$$C_K(u) := \{u(1), \ldots, u(1) + K - 1\} \times \{u(2) - K + 1, \ldots, u(2)\} \,.$$

We say that the box $C_K(u)$ is good if for all $v \in C_K(u)$, $W_v = 1$ and v is open.

Remark 6.1. *We point out that when $C_K(u)$ is good then there are no paths crossing it and touching either $(-\infty, u(1)] \times \{u(2)\}$ or $[u(1) + K, \infty) \times \{u(2)\}$.*

Now let us define the following random variables

$$g_K^+(u) := \inf\{n \geq 1; C_K(u + (n-1)Ke_1) \text{ is good}\},$$

and

$$g_K^-(u) := \inf\{n \geq 1; C_K(u - (nK-1)e_1) \text{ is good}\}.$$

Therefore $C_K\big(u + Kg_K^+(u)e_1\big)$ is the first translation of $C_K(u)$ to the right of u by multiples of K that is good and $C_K\big(u - Kg_K^-(e_1)\big)$ is the first translation of $C_K(u)$ to the left of u by multiples of K that is good.

The first Lemma below allow us to consider the counting variables $\eta_{\mathcal{X}_n}(t_0, t, a, b)$ only on integer starting times t_0.

Lemma 6.1. *Take $a < b \in \mathbb{R}$, $\mathcal{X}_n$ as defined in (2.2) with γ and σ as in Proposition 5.1 and $\eta_{\mathcal{X}_n}(t_0, t, a, b)$ as in the Theorem 2.2. Then for all $\epsilon > 0$ there exits a constant M_ϵ, not depending on a, b, γ and σ, such that*

$$\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0, t, a, b)| > 1\big] \leq \mathbb{P}\big[|\eta_{\mathcal{X}}(0, n^2\gamma t, n\sigma a - M_\epsilon, n\sigma b + M_\epsilon)| > 1\big] + \epsilon$$

for all $t_0 \in \mathbb{R}, t > 0$ and $n \geq 1$.

Proof. Note that any path that cross $[n\sigma a, n\sigma b] \times \{n^2\gamma t_0\}$ also cross the interval

$$\Big[n\sigma a - Kg_K^-\big((\lfloor n\sigma a\rfloor, \lfloor n^2\gamma t_0\rfloor + 1)\big), n\sigma b + Kg_K^+\big((\lfloor n\sigma b\rfloor + 1, \lfloor n^2\gamma t_0\rfloor + 1)\big)\Big] \times \{\lfloor n^2\gamma t_0\rfloor + 1\}.$$

Then

$$\begin{aligned}
&\mathbb{P}[|\eta_{\mathcal{X}_n}(t_0, t, a, b)| > 1] \\
&= \mathbb{P}\Big[|\eta_{\mathcal{X}}(n^2\gamma t_0, n^2\gamma t, n\sigma a, n\gamma b)| > 1\Big] \\
&\leq \mathbb{P}\Big[\big|\eta_{\mathcal{X}}(\lfloor n^2\gamma t_0\rfloor), n^2\gamma t, n\sigma a - Kg^-\big((\lfloor n\sigma a\rfloor, \lfloor n^2\gamma t_0\rfloor + 1)\big), n\sigma b + Kg^+\big((\lfloor n\sigma b\rfloor + 1, \lfloor n^2\gamma t_0\rfloor + 1)\big)\big| > 1\Big].
\end{aligned}$$

Take M_ϵ large enough such that

$$\mathbb{P}\Big[Kg^-\big((\lfloor n\sigma a\rfloor, \lfloor n^2\gamma t_0\rfloor + 1)\big) > M_\epsilon] = \mathbb{P}\Big[Kg^+\big((\lfloor n\sigma b\rfloor + 1, \lfloor n^2\gamma t_0\rfloor + 1)\big) > M_\epsilon\Big] \leq \frac{\epsilon}{2}.$$

Then by the translation invariant we get

$$\begin{aligned}
\mathbb{P}[|\eta_{\mathcal{X}_n}(t_0, t, a, b)| > 1] &\leq \mathbb{P}\big[|\eta_{\mathcal{X}}(\lfloor n^2\gamma t_0\rfloor, n^2\gamma t, n\sigma a - M_\epsilon, n\sigma b + M_\epsilon)| > 1\big] + \epsilon \\
&= \mathbb{P}\big[|\eta_{\mathcal{X}}(0, n^2\gamma t, n\sigma a - M_\epsilon, n\sigma b + M_\epsilon)| > 1\big] + \epsilon.
\end{aligned}$$

□

Our next result says that the number of paths in $\mathcal{X}$ starting before time t that cross a finite length interval at time t have finite absolute moment of any order.

Lemma 6.2. *Let us define $\mathcal{X}^{t^-}$ as the set of paths in $\mathcal{X}$ that start before or at time t and by $X^{t-}(t)$ its values on time t. Then we have that*

$$\mathbb{E}\Big[\big|\mathcal{X}^{t^-}(t) \cap [a, b]\big|^k\Big] < \infty$$

for $a < b \in \mathbb{R}$ and $k \geq 1$.

Proof. We will assume that $t = a = 0$ and $b = 1$. The general case is analogous. For $j \in \mathbb{Z}$ let us define $\zeta_j := \inf\big\{n \geq 0; \sum_{i=0}^{n} {}_{\{(j,-i) \text{ is open }\}} = K\big\}$ and the random region D as

$$D := \big\{v \in \mathbb{Z}^2; -Kg_K^-((0,0)) \leq v(1) \leq 1 + Kg_K^+((0,0)) \text{ and } -\zeta_{v(1)} \leq v(2) \leq 0\big\}.$$

In next figure we show a possible face of D.

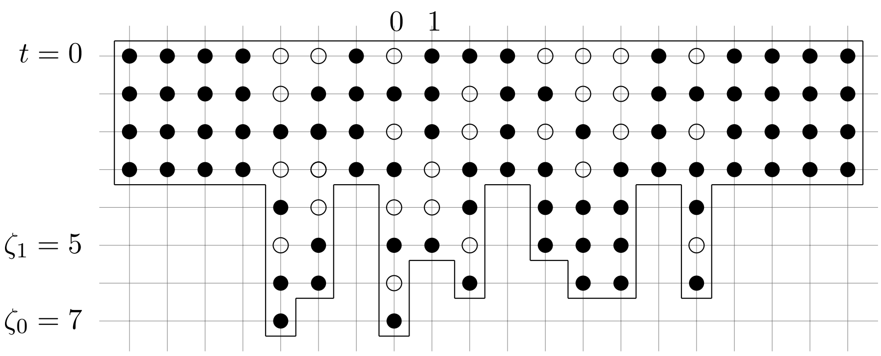

FIGURE 8. In this picture we assume that $K = 4$. The blacks balls represent open points and the white ones represent closed points. The region D is given by the set of sites inside the contour in bold. Note that $g_4^+ = 3$ and $g_4^- = 2$.

Note that there is no paths crossing $[0,1] \times \{0\}$ without landing in D, hence

$$\left|\mathcal{X}^{0^-}(0) \cap [0,1]\right| \leq \left|D\right| = \sum_{j=1}^{Kg^+((1,0))} \zeta_j + \sum_{j=0}^{Kg^-((0,0))} \zeta_{-j}$$

Now using the Lemma A.1 we have that $\mathbb{E}\Big[\left|\mathcal{X}^{0^-}(0) \cap [0,1]\right|^k\Big] < \infty$ for all $k \geq 1$. □

We are going to need another result about renewal times. Here we need to define the renewal times for a finite collection of paths in $\mathcal{X}^{t^-}$ such that all we know about them is that they cross an interval $[a,b]$ at time t. Therefore we are interested in $\{\pi \in \mathcal{X}^{t^-} : \pi_(t) \in [a,b]\}$ which is almost surely finite by Lemma 6.2 since it is the set of paths in $\mathcal{X}^{t^-}$ whose projection at time t is in $\mathcal{X}^{t^-}(t) \cap [a,b]$. The proof is analogous to the proof of Proposition 3.1 and it will be omitted.

Lemma 6.3. *Fix $a < b$ and consider the collection of paths $\Gamma = \{\pi \in \mathcal{X}^{t^-} : \pi(t) \in [a,b]\}$. Then there exist random variables T_0 and Z_0 such that*

(i) *$t < T_0$ and $T_0 - t \leq Z_0$.*
(ii) *T_0 is a common renewal time for all paths in Γ.*
(iii) *$\sup_{\pi\in\Gamma} \sup_{t\leq s\leq T_0} |\pi(s) - \pi(t)| \leq Z_0$.*
(iv) *For all $k \in \mathbb{N}$ we have that $\mathbb{E}\big[(Z_0)^k\big] < \infty$.*

We need one more result before we prove condition B of Theorem 2.2.

Lemma 6.4. *There exists a constant $C_1 > 0$ such that*

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,m)| > 1\big] \leq \frac{C_1 m}{\sqrt{k}}\,,$$

for every $m \geq 1$.

Proof. Note that

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,m)| > 1\big] \leq \sum_{i=1}^{m} \mathbb{P}\big[|\eta_{\mathcal{X}(0,k,i-1,i)}| > 1\big] = m\mathbb{P}\big[|\eta_{\mathcal{X}(0,k,0,1)}| > 1\big],$$

and

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,1)| > 1\big] = \sum_{j=2}^{\infty} \mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,1)| > 1\big||\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big]\mathbb{P}\big[||\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big] \tag{6.1}$$

Given $|\mathcal{X}^{0^-}(0) \cap [0,1]| = j$, let $\pi_1, \ldots, \pi_j$ be the paths in $\mathcal{X}^{0^-}$ such that $\pi_i(0) \in [0,1]$ for $i = 1, \ldots, j$ and define

$$\nu_{i,i+1} := \inf\{n \geq 1 : \pi_i(t) = \pi_{i+1}(t), \text{ for all } t \geq n\}.$$

See that

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,1)| > 1\big||\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big] \leq \sum_{i=1}^{j-1} \mathbb{P}\big[|\nu_{i,i+1} > k\big||\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big]\,. \tag{6.2}$$

Let T_0 and Z_0 be random variables as in Lemma 6.3. Then for $i = 1, \dots, j$ we have that

$$\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \mathbb{P}\Big[\nu_{i,i+1} > k, T_0 > \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] + \mathbb{P}\Big[\nu_{i,i+1} > k, T_0 \le \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\le \mathbb{P}\Big[T_0 > \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] + \mathbb{P}\Big[\nu_{i,i+1} > k, T_0 \le \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\le \frac{2\mathbb{E}\Big[T_0 \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]}{k} + \mathbb{P}\Big[\nu_{i,i+1} > k, T_0 \le \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big].
\end{aligned} \tag{6.3}$$

Now define $\nu^{T_0}_{i,i+1} := \inf\{t \ge T_0; \pi_i(s) = \pi_{i+1}(s) \text{ for all } s \ge t\}$. Then we have that

$$\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k, T_0 \le \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \sum_{l=1}^{\infty} \mathbb{P}\Big[\nu_{i,i+1} > k, T_0 \le \frac{k}{2}, \big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\le \sum_{l=1}^{\infty} \mathbb{P}\Big[\nu^{T_0}_{i,i+1} > \frac{k}{2}, \big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \sum_{l=1}^{\infty} \mathbb{P}\Big[\nu^{T_0}_{i,i+1} > \frac{k}{2} \Big| \big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| = l\Big] \mathbb{P}\Big[\big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big].
\end{aligned} \tag{6.4}$$

By Collorary 4.1 we have that

$$\mathbb{P}\Big[\nu^{T_0}_{i,i+1} > \frac{k}{2} \Big| \big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| = l\Big] \le \frac{2lC}{\sqrt{k}}. \tag{6.5}$$

Replacing (6.5) in (6.4) we get

$$\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k, T_0 \le \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\le \sum_{l \ge 1} \frac{2lC}{\sqrt{k}} \mathbb{P}\Big[\big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \frac{2C}{\sqrt{k}} \mathbb{E}\Big[\big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \qquad (6.6) \\
& \qquad (6.7)
\end{aligned}$$

Since

$$\big|\pi_i(T_0) - \pi_{i+1}(T_0)\big| \le \big|\pi_i(T_0) - \pi_i(0)\big| + \big|\pi_i(0) - \pi_{i+1}(0)\big| + \big|\pi_{i+1}(0) - \pi_{i+1}(T_0)\big| \le 2Z_0 + 1\,,$$

we have that (6.6) is bounded above by

$$\frac{2C}{\sqrt{k}} \mathbb{E}\Big[2Z_0 + 1 \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]. \tag{6.8}$$

Now replacing (6.8) in (6.3) we obtain,

$$\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k\Big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\Big] \\
&\leq \frac{2\mathbb{E}\Big[Z_0\Big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\Big]}{k} + \frac{2C}{\sqrt{k}}\mathbb{E}\Big[2Z_0+1\Big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\Big] \\
&\leq \frac{2(1+C)\mathbb{E}\Big[2Z_0+1\Big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\Big]}{\sqrt{k}}. \qquad (6.9)
\end{aligned}$$

Hence by (6.2) and (6.9),

$$\begin{aligned}
&\mathbb{P}\Big[\big|\eta_{\mathcal{X}}(0,k,0,1)\big| > 1\Big||\mathcal{X}^{0^-}(0)\cap[0,1]|j\Big] \\
&\qquad\leq \frac{2(1+C)}{\sqrt{k}}\, j\,\mathbb{E}\Big[2Z_0+1\Big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\Big]. \qquad (6.10)
\end{aligned}$$

Replacing (6.10) in (6.1) we get that $\mathbb{P}[|\eta_{\mathcal{X}}(0,k,0,1)| > 1]$ is dominated by

$$\frac{2(1+C)}{\sqrt{k}}\sum_{j=2}^{\infty} j\mathbb{E}\big[2Z_0+1\big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big] \qquad (6.11)$$

Note that

$$\begin{aligned}
&\sum_{j=2}^{\infty} j\mathbb{E}\big[2Z_0+1\big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big] \\
&\leq \Big(\sum_{j=2}^{\infty} j^2\mathbb{P}\big[||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]\Big)^{\frac{1}{2}}\Big(\sum_{j=2}^{\infty}\mathbb{E}\big[2Z_0+1\big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]^2\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{j=2}^{\infty} j^2\mathbb{P}\big[||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]\Big)^{\frac{1}{2}}\Big(\sum_{j=2}^{\infty}\mathbb{E}\big[(2Z_0+1)^2\big||\mathcal{X}^{0^-}(0)\cap[0,1]| = j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|\big]\Big)^{\frac{1}{2}} \\
&= \mathbb{E}\Big[\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|^2\Big]^{\frac{1}{2}}\mathbb{E}\big[(2Z_0+1)^2\big]^{\frac{1}{2}}. \qquad (6.12)
\end{aligned}$$

Take $C_1 := 2(1+C)\mathbb{E}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|^2\big]^{\frac{1}{2}}\mathbb{E}\big[(2Z_0+1)^2\big]^{\frac{1}{2}}$ which is finite by Lemma 6.2 and Lemma 6.3. Replacing (6.12) in (6.11) we have that

$$\mathbb{P}[|\eta_{\mathcal{X}}(0,k,0,1)| > 1] \leq \frac{C_1}{\sqrt{k}},$$

which finishes the proof. □

Proof of the condition B of the Theorem 2.2. Fix $\epsilon > 0$ and take M_ϵ as in the statement of Lemma 6.1, from that result we get that

$$\sup_{t_0,a\in\mathbb{R}}\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0,t,a-\epsilon,a+\epsilon)| > 1\big]$$

is bounded above by

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,n^2\gamma t, n\sigma(a-\epsilon)-M_\epsilon, n\sigma(a+\epsilon)+M_\epsilon)| > 1\big] + \epsilon\,.$$

Then by Lemma 6.4

$$\sup_{t>\beta}\sup_{t_0,a\in\mathbb{R}}\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0,t,a-\epsilon,a+\epsilon)| > 1\big] \leq \frac{C_1}{n\sqrt{\gamma\beta}}2(n\sigma\epsilon+M_\epsilon)+\epsilon.$$

Hence

$$\limsup_{n\to\infty} \sup_{t>\beta} \sup_{t_0,a\in\mathbb{R}} \mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0,t,a-\epsilon,a+\epsilon)|>1\big] \leq \Big(\frac{2C_1\sigma}{\sqrt{\beta\gamma}}+1\Big)\epsilon \to 0 \text{ as } \epsilon\to 0^+.$$

So we have condition B. □

7. The condition E

Following [NRS05], the verification of condition E is a consequence of Lemmas 7.1 and 7.2 which are versions of respectively Lemmas 6.2 and 6.3 in that paper. Lemmas 7.1 follows from Lemma 7.4 below as Lemma 6.2 follows from Lemma 6.4 in [NRS05]. The proof of the version of Lemma 7.2 made in [NRS05] for the Nonsimple Random Walk could be adapted for our process.

We start making some definitions. For a set of paths $Y\subset\Pi$ define

(i) Y^{s^-} := the subset of paths in Y such that start before or at time s;
(ii) For $A\subset\mathbb{R}$ define $Y^{s^-,A} := \{\pi\in Y^{s-};\pi(s)\in A\}$;
(iii) For $s\leq t$ and $A\subset\mathbb{R}$ define $Y^{s-}(t) := \{\pi(t);\pi\in Y^{s-}\}$ and $Y^{s-,A}(t) := \{\pi(t);\pi\in Y^{s-,A}\}$.

Lemma 7.1. *Let Z_{t_0} be any subsequential limit of $\{\mathcal{X}_n^{t_0^-}\}$. For any $\epsilon>0$, $Z_{t_0}(t_0+\epsilon)$ is almost surely locally finite and*

$$\mathbb{E}\Big[\big|Z_{t_0}(t_0+\epsilon)\cap(a,b)\big|\Big] \leq \frac{(b-a)C_4}{\sqrt{\epsilon}}.$$

Lemma 7.2. *Let Z_{t_0} be any subsequential limit of $\{\mathcal{X}_n^{t_0^-}\}$ and $\epsilon>0$. Denote by $\mathcal{Z}_{t_0}^{(t_0+\epsilon)T}$ the set of paths in Z_{t_0} that start before time t truncated before time $t_0+\epsilon$. Then $\mathcal{Z}_{t_0}^{(t_0+\epsilon)T}$ is distributed as coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0+\epsilon)\subset\mathbb{R}^2$.*

As pointed out above, for the proof of Proposition 7.2 see Lemma 6.3 in [NRS05]. The remain of the section is devoted to state and prove Lemma 7.4 below, but we first need the following result:

Lemma 7.3. *There exists a constant C_2 such that*

$$\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(t)\cap[0,1)\big|\Big] \leq \frac{C_2}{\sqrt{t}}$$

for all $t>0$.

Proof. Fix $M\in\mathbb{Z}_+$ arbitrarily and note that

$$\begin{aligned} M\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(t)\cap[0,1)\big|\Big] &= \mathbb{E}\Big[\big|\mathcal{X}^{0^-}(t)\cap[0,M)\big|\Big] \\ &= \sum_{i\in\mathbb{Z}}\mathbb{E}\Big[\big|\mathcal{X}^{0^-,[iM,(i+1)M)}(t)\cap[0,M)\big|\Big] \\ &= \sum_{i\in\mathbb{Z}}\mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,M)}(t)\cap[iM,(i+1)M)\big|\Big] = \mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,M)}(t)\big|\Big], \end{aligned}$$

where the third equality above follows from the symmetry of the GRDF paths. Since $\mathcal{X}^{0^-,[0,M)}(0)$ has at least M points which are 0,1,2,...,M-1, then

$$
\begin{aligned}
&M\mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t)\cap[0,1)\Big|\Big]\\
&\qquad=\sum_{j=M}^{\infty}\mathbb{E}\Big[|\mathcal{X}^{0^-,[0,M)}(t)|\Big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]\mathbb{P}\Big[|\mathcal{X}^{0^-,[0,M)}(0)|=j\Big].
\end{aligned}
$$

From here the proof is very close to that of Lemma 6.4, given $|\mathcal{X}^{0^-}(0)\cap[0,M)|=j$, $j\geq M$, let $\pi_1,\dots,\pi_j$ be the paths in $\mathcal{X}^{0^-}$ such that $0\leq\pi_1(0)<\pi_2(0)<...<\pi_j(0)<M$ for $i=1,\dots,j$ and define

$$\nu_{i,i+1}:=\inf\{n\geq 1:\pi_i(t)=\pi_{i+1}(t),\ \text{for all } t\geq n\}.$$

Then

$$\mathbb{E}\Big[|\mathcal{X}^{0^-,[0,M)}(t)|\Big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]\leq\mathbb{E}\Big[1+\sum_{i=1}^{j-1}1_{\{\nu_{i,i+1}>t\}}\Big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big].$$

Note that $|\pi_i(0)=\pi_{i+1}(0)|\leq 1$ because $0,1,...,M-1\in\mathcal{X}^{0^-,[0,M)}(0)$, then by Lemma 6.3 and (6.9) we have that there exists a constant $C>0$ and a integrable random variable Z_0, both not depending on M, such that

$$
\begin{aligned}
P\Big[\nu_{i,i+1}>t\Big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]&\leq\frac{2(1+C)}{\sqrt{t}}\mathbb{E}\Big[2Z_0+1\big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]\\
&\leq\frac{2(1+C)}{\sqrt{t}}\mathbb{E}\Big[3Z_0\big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]\\
&=\frac{\widetilde{C}}{\sqrt{t}}\mathbb{E}\Big[Z_0\big||\mathcal{X}^{0^-,[0,M)}(0)|=j\Big].
\end{aligned}
$$

Hence

$$M\mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t)\cap[0,1)\Big|\Big]\leq 1+\frac{\widetilde{C}}{\sqrt{t}}\sum_{j=M}^{\infty}j\mathbb{E}\Big[Z_0\big|\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]\mathbb{P}\Big[|\mathcal{X}^{0^-,[0,M)}(0)|=j\Big]$$

which as in (6.12) can be shown to be bounded above by

$$
\begin{aligned}
&1+\frac{\widetilde{C}}{\sqrt{t}}\Big(\mathbb{E}\Big[|\mathcal{X}^{0^-,[0,M)}(0)|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[(Z_0)^2]\Big)^{\frac{1}{2}}\\
&\qquad\leq 1+\frac{\widetilde{C}}{\sqrt{t}}\Big(M\mathbb{E}\Big[\sum_{i=1}^{M}|\mathcal{X}^{0^-,[i-1,i]}(0)|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[(Z_0)^2]\Big)^{\frac{1}{2}}\\
&\qquad=1+\frac{\widetilde{C}}{\sqrt{t}}M\Big(\mathbb{E}\Big[|\mathcal{X}^{0^-,[0,1]}(0)|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[(Z_0)^2]\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Thus

$$\mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t)\cap[0,1)\Big|\Big]\leq\frac{1}{M}+\frac{C_2}{\sqrt{t}},$$

where $C_2:=\widetilde{C}\Big(\mathbb{E}[(Z_0)^2]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[|\mathcal{X}^{0^-,[0,1]}(0)|^2\Big]\Big)^{\frac{1}{2}}$ which is finite by Lemma 6.2. Since M is arbitrary we obtain the bound in the statement. $\square$

Lemma 7.4. *There exists a constant C_3, independent of M, such that*

$$\mathbb{E}\Big[\big|\mathcal{X}_n^{0^-}(t)\cap[0,M)\big|\Big] \le \frac{MC_3}{\sqrt{t}}\,,$$

for every $n\ge 1$ and $M\ge 1$.

Proof. Using Lemma 7.3 we have that for all $n\ge 1$

$$\begin{aligned}\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(n^2\gamma t)\cap[0,n\sigma M)\big|\Big] &= n\sigma M\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(n^2\gamma t)\cap[0,1)\big|\Big]\\ &\le \frac{n\sigma MC_2}{\sqrt{n^2\gamma t}} = \frac{\gamma^{-1/2}\sigma MC_2}{\sqrt{t}}.\end{aligned}$$

□

8. The condition T

In this section we will prove the condition T in Theorem 2.2 which follows from Proposition 8.1 in the end of this section. The idea behind the proof comes from [NRS05]. Technical details related to the renewals times impose an extra difficult - not much really - to the proof given in [NRS05]. Even though the proof is very similar, we present it here for the sake of completeness.

Recall the definitions from the statement of condition T in Section 2. By homogeneity of the GRDF all the estimates on $A_{\mathcal{X}_n}(x_0,t_0;\rho,t)$ are uniform on $(x_0,t_0)\in\mathbb{R}^2$. Here we only consider $(x_0,t_0)=(0,0)$ leaving the verification for other choices of t_0 to the reader. The case $n\gamma t_0\notin\mathbb{Z}$ demands an extra care, but can be dealt analogously as done the previous sections to deal with paths crossing some time level not necessarily on the rescaled space/time lattice. With this in mind, condition T is a consequence of the next result.

Proposition 8.1. *Denote by $A^+_{\mathcal{X}_n}(x_0,t_0;\rho,t)$ the event that $\mathcal{X}_n$ contains a path touching both $R(x_0,t_0;\rho,t)$ and the right boundary of the rectangle $R(x_0,t_0;20\rho,4t)$. Then*

$$\lim_{t\to 0^+}\frac{1}{t}\limsup_{n\to\infty}\mathbb{P}\Big[A^+_{\mathcal{X}_n}(0,0;\rho,t)\Big]=0.$$

Before we prove Proposition 8.1 we need some lemmas whose proofs will be postponed to Appendix C. The first Lemma gives an uniform bound on the overshoot distribution on the renewal times for paths in the GRDF.

Lemma 8.1. *For $x\in\mathbb{Z}_-$ let $(T_i^x)_{i\ge 0}$ be a sequence of renewal times of the paths $\pi^{(x,0)}$ such that there exist a sequence of i.i.d. random variables $(Z_i^x)_{i\ge1}$ with the following property*

$$\Big|\pi^x(T_i^x)(1)-\pi^x(T_{i-1}^x)(1)\Big|\le Z_i^x \text{ for all } i\ge 1,$$

and $\mathbb{E}\big[(Z_1^x)^{k+2}\big]<\infty$ for a fixed $k\ge1$. Define Y_i^x, $i\ge0$, as the first component of the path $\pi^{(x,0)}$ on the random time T_i^x. Also define $\nu_+^x:=\inf\{n\ge1;Y_n^x\ge1\}$. Then we have that

$$\sup_{x\in\mathbb{Z}_-}\mathbb{E}\big[(Y^x_{\nu_+^x})^k\big]<\infty.$$

A path in the GRDF is obtained from linear interpolation between open points in $\mathbb{Z}^2$, we say that these open points defining the path are the ones visited by the path. The next Lemma states that the probability of having paths that cross a box $R(0,0;n\rho\sigma,n^2t\gamma)$ but do not visit any point in $R(0,0;2n\rho\sigma,2n^2t\gamma)$ goes to zero as $n\to\infty$.

Lemma 8.2. *Let $D(n\rho\sigma, n^2t\gamma)$ be the event that paths in $\mathcal{X}$ cross $R(0,0;n\rho\sigma,n^2t\gamma)$ without visit any point in $R(0,0;2n\rho\sigma,2n^2t\gamma)$, then*

$$\lim_{n\to\infty}\mathbb{P}\Big[D(n\rho\sigma, n^2t\gamma)\Big]=0.$$

Lemma 8.3. *Let $x,y,x_1,\dots,x_m$ be points in $\mathbb{Z}$ with $x<y$. Define $u=(x,0)$ and $v=(y,0)$. Consider the random times $(T_n)_{n\geq 1},(\tau_n(u))_{n\geq 1}$ and $(\tau_n(v))_{n\geq 1}$ as introduced in Corollary 3.1 for the points $(x_1,0)$,…,$(x_m,0)$,u and v. Put $T_0=\tau_0(u)=\tau_0(v)=0$ and let $\widetilde{\pi}^u$ and $\widetilde{\pi}^v$ be the linear interpolations of $(X_{\tau_n(u)}(u))_{n\geq 0}$ and $(X_{\tau_n(v)}(v))_{n\geq 0}$ respectively. Then for $\rho>0$ and the stopping time*

$$\nu_{x,y,\rho^+}:=\inf\{s\geq 0;\widetilde{\pi}^x(s)-\widetilde{\pi}^y(s)\geq n\rho\sigma\},$$

there exists a constant $C(t,\rho)$ depending only on t and ρ such that for all n large enough we have that

$$\mathbb{P}\big[\nu_{x,y,\rho^+}<\nu_{x,y}\wedge(n^2t\gamma)\big]<\frac{C(t,\rho)}{n}$$

where $\nu_{x,y}$ is the first time that $\widetilde{\pi}^x$ and $\widetilde{\pi}^y$ coalesce.

Remark 8.1. *The paths $\widetilde{\pi}^u$ and $\widetilde{\pi}^v$ in the statement of Lemma 8.3 are not paths of the GRDF. They are obtained from linear interpolation only on the points visited by the GRDF paths on the renewal times.*

Proof of Proposition 8.1. Let π_1,π_2,π_3,π_4 be the paths that start in $5\lfloor n\rho\sigma\rfloor, 9\lfloor n\rho\sigma\rfloor, 13\lfloor n\rho\sigma\rfloor$ and $17\lfloor n\rho\sigma\rfloor$ respectively at time zero. Let us denote the event that π_i stay within a distance $n\rho\sigma$ of $\pi_i(0)$ until time $2tn^2\gamma$ by $B_i^{n,t}$ for $i=1,\dots,4$, see Figure 9 below. From the invariance principle we have that

$$\lim_n\mathbb{P}[(B_i^{n,t})^c]=\mathbb{P}[\sup_{s\in[0,t]}|\mathcal{B}_s|>\rho]\leq 4e^{-\frac{\rho^2}{2t}}$$

for all $i=1,\dots,4$. Then

$$\frac{1}{t}\lim_{n\to\infty}\mathbb{P}\Big[(B_i^{n,t})^c\Big]\to 0 \text{ as } t\to 0^+. \tag{8.1}$$

See that

$$\begin{aligned}\lim_{t\to 0^+}\frac{1}{t}\limsup_{n\to\infty}\mathbb{P}\Big[A^+_{\mathcal{X}_n}(0,0;\rho,t)\Big] &= \lim_{t\to 0^+}\frac{1}{t}\limsup_{n\to\infty}\mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma,tn^2\gamma)\Big]\\ &\leq \lim_{t\to 0^+}\frac{1}{t}\limsup_{n\to\infty}\mathbb{P}\Big[D(n\rho\sigma,tn^2\gamma)\Big]+4\lim_{t\to 0^+}\frac{1}{t}\lim_{n\to\infty}\mathbb{P}\Big[(B_1^{n,t})^c\Big]+\\ &+\lim_{t\to 0^+}\frac{1}{t}\limsup_{n\to\infty}\mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma,tn^2\gamma),\cap_{i=1}^4B_i^{n,t},\big(D(n\rho\sigma,tn^2\gamma)\big)^c\Big].\end{aligned}$$

By Lemma 8.2 we have that

$$\limsup_{n\to\infty}\mathbb{P}\Big[D(n\rho\sigma,tn^2\gamma)\Big]=0\,,\quad\text{for every } t>0$$

and by (8.1)

$$\lim_{t\to 0^+}\frac{1}{t}\lim_{n\to\infty}\mathbb{P}\Big[(B_1^{n,t})^c\Big]=0\,.$$

So we only have to prove, see Figure 9, that for every $t>0$

$$\limsup_{n\to\infty}\mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma,tn^2\gamma),\cap_{i=1}^4B_i^{n,t},(D(n\rho\sigma,tn^2\gamma))^c\Big]=0\,. \tag{8.2}$$

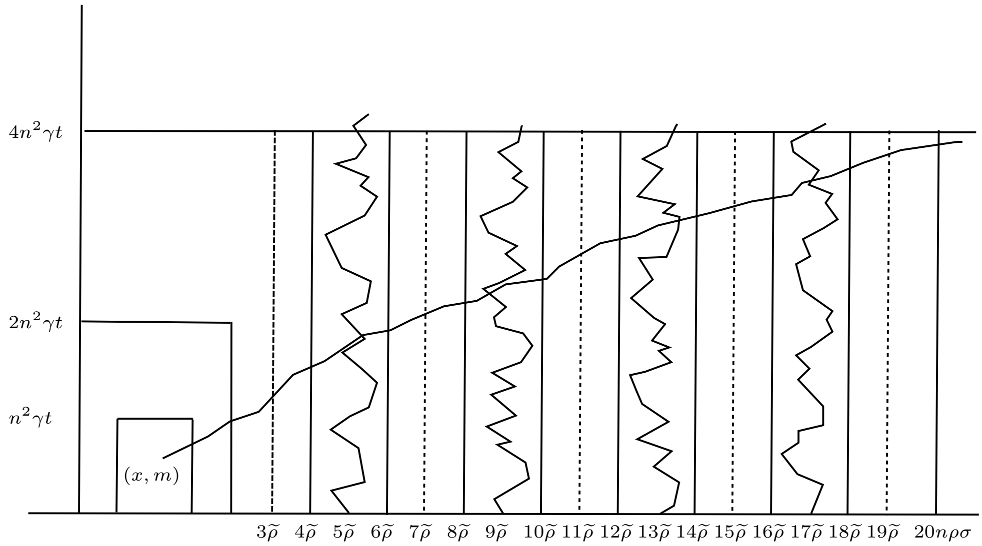

FIGURE 9. Realization of $A^{+}_{\mathcal{X}}(0,0;\rho n\sigma,tn^2\gamma)\cap\cap_{i=1}^4 B_i^{n,t}$ where $A^{+}_{\mathcal{X}}(0,0;\rho n\sigma,tn^2\gamma)$ occurs because the path $\pi^{(x,m)}$, for some $(x,m)\in R(0,0;\rho n\sigma,tn^2\gamma)$, touchs the right boundary of the rectangle $R(0,0;20\rho n\sigma,4tn^2\gamma)$. Notation: $\tilde\rho=\lfloor n\sigma\rho\rfloor$.

Fix some $(x,m)\in R(0,0;2n\rho\sigma,2n^2t\gamma)$ and take $(T_i)_{i\geq 1}$ renewal times introduced in the Corollary 3.2 for the points $(5\lfloor n\rho\sigma\rfloor,0)$,$(9\lfloor n\rho\sigma\rfloor,0)$,$(13\lfloor n\rho\sigma\rfloor,0)$, $(17\lfloor n\rho\sigma\rfloor,0)$ and the (x,m). We will denote by $\big(Y_i^{(x,m)}\big)_{i\geq 0}$ the random walks built as the first component of the path $\pi^{(x,m)}$ on the renewal times $(T_i)_{i\geq 1}$. Take the stopping times $\nu_j^{(x,m)}$ for $j=1,\dots,5$ as the first time that $\big(Y_i^{(x,m)}\big)_{i\geq 0}$ exceeds $(4j-1)\lfloor n\rho\sigma\rfloor$; and $\nu^{(x,m)}$ the first time that $\pi^{(x,m)}$ exceeds $20n\rho\sigma$. Then

$$\mathbb{P}\big[\nu^{(x,m)}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\big]\leq\mathbb{P}\big[T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\big]+\mathbb{P}\big[\nu^{(x,m)}<4n^2t\gamma,T_{\nu_5^{(x,m)}}\geq 4n^2t\gamma\big].\quad(8.3)$$

Note that on the event $\big\{\nu^{(x,m)}<4n^2\gamma t,T_{\nu_5^{(x,m)}}\geq 4n^2t\gamma\big\}$ the path $\pi^{(x,m)}$ cross the interval $\big(19\lfloor n\rho\sigma\rfloor,20n\rho\sigma\big)$ without renewal before time $n^2t\gamma$. Because the displacement between consecutive renewal times is bounded by some random variable Z with finite moments, and up to time $n^2\gamma t$ the number of renewals is bounded by $n^2\gamma t$ we have that

$$\mathbb{P}\big[\nu^{(x,m)}<4n^2\gamma t,T_{\nu_5^{(x,m)}}\geq 4n^2t\gamma\big]\leq n^2t\gamma\mathbb{P}\big[Z>n\rho\sigma\big]\leq\frac{n^2t\gamma\mathbb{E}[Z^6]}{(n\rho\sigma)^6}\leq\frac{C_1}{n^4}.\quad(8.4)$$

We also have

$$\begin{aligned}&\mathbb{P}\Big[T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]\\&\leq\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}\leq\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor,\ j=1,...5,\ T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]+\sum_{j=1}^5\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}>\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor\Big]\\&\leq\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}\leq\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor,\ j=1,...5,\ T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]+5\sup_{x\in\mathbb{Z}_-}\mathbb{P}\Big[Y^{(x,m)}_{\nu_+^x}>\frac{\lfloor n\rho\sigma\rfloor}{2}\Big].\quad(8.5)\end{aligned}$$

By the Lemma 8.1 and Corollary 3.2 there exists a constant C_2 such that

$$\sup_{x\in\mathbb{Z}_-}\mathbb{P}\Big[Y^{(x,m)}_{\nu_+^x}>\frac{\lfloor n\rho\sigma\rfloor}{2}\Big]\leq\frac{C_2}{n^4}.$$

Using the strong Markov property and Lemma 8.3 we get a constant C_3 such that

$$\begin{aligned}&\mathbb{P}\Big[Y^{(x,m)}_{\nu^{(x,m)}_j} \leq \big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor,\ j=1,...5,\ T_{\nu^{(x,m)}_5} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\Big]\\ &\leq \mathbb{P}\Big[\nu^+_{x,y,\rho} < \nu_{x,y}\wedge(n^2t\gamma)\Big]^4 \leq \frac{C_3}{n^4}.\end{aligned}$$

Hence by (8.5)

$$\mathbb{P}\Big[T_{\nu^{(x,m)}_5} < 4n^2t\gamma, \cap_i^4 B_i^{n,t}\Big] \leq \frac{C_3}{n^4} + \frac{5C_2}{n^4}. \tag{8.6}$$

Now we can go back to (8.3), use (8.4), (8.5) and (8.6) to conclude that

$$\mathbb{P}\big[\nu^{(x,m)} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\big] \leq \frac{(5C_1+C_2+C_3)}{n^4}.$$

Therefore we can estimate the probability in (8.2) as

$$\begin{aligned}&\mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma), \cap_{i=1}^4 B_i^{n,t}, \{D(n\rho\sigma, n^2t\gamma)\}^c\Big]\\ &\leq \mathbb{P}\Big[\exists (x,m)\in R(0,0;2n\rho\sigma, 2n^2t\gamma);\, \nu^{(x,m)} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\Big].\end{aligned}$$

Since $R(0,0;2n\rho\sigma,2n^2t\gamma)$ has $8t\rho\sigma\gamma n^3$ points we have that

$$\begin{aligned}&\limsup_{n\to\infty}\mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma), \cap_{i=1}^4 B_i^{n,t}, \{D(n\rho\sigma, n^2t\gamma)\}^c\Big]\\ &\leq \limsup_{n\to\infty}(8t\rho\sigma\gamma n^3)\mathbb{P}\big[\nu^{(x,m)} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\big]\\ &\leq \limsup_{n\to\infty}(8t\rho\sigma\gamma n^3)\frac{(5C_1+C_2+C_3)}{n^4} = 0.\end{aligned}$$

□

Appendix A. Well posedness

In this section we will see that $\overline{\mathcal{X}}_n$, the closure of $\mathcal{X}_n$, is a compact set in (Π, d) for all $n \geq 1$. Therefore we are indeed working with random elements of $(\mathcal{H}, d_{\mathcal{H}})$ where the Brownian web is defined. Other result we present here is that any path in $\overline{\mathcal{X}}_n$ coincide locally with a path in $\mathcal{X}_n$. This is useful when we verify the conditions of the Theorem 2.2 because we can work with $\mathcal{X}_n$ instead of $\overline{\mathcal{X}}_n$. We start stating and proving some lemma.

Lemma A.1. *Let N be some positive integer random variable and $(\zeta_n)_{n\geq 1}$ a non-negative sequence of identically distributed random variables. If for some $k \geq 1, \delta > 0$ and $l > \frac{(k+2)(1+\delta)}{\delta}$ we have $\mathbb{E}[\zeta_1^{k(1+\delta)}]$ and $\mathbb{E}[N^l]$ finite, then for $S := \sum_{n=1}^N \zeta_n$ we get that $\mathbb{E}[S^k]$ is also finite.*

Proof. We have that $0 \leq S \leq N\max_{1\leq j\leq N}\zeta_j$ what implies that $S^k \leq N^k \max_{1\leq j\leq N}\zeta_j^k \leq N^k\sum_{j=1}^N \zeta_j^k$. Hence

$$\mathbb{E}\big[S^k\big] \leq \mathbb{E}\Big[N^k\sum_{j=1}^N \zeta_j^k\Big] = \sum_{n=1}^\infty n^k \sum_{j=1}^n \mathbb{E}\big[\ _{\{N=n\}}\zeta_j^k\big].$$

Applying Cauchy-Schwartz inequality we get

$$\mathbb{E}\big[S^k\big] \leq \sum_{n=1}^{\infty} n^k \sum_{j=1}^{n} \mathbb{E}\big[\zeta_j^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \mathbb{P}[N=n]^{\frac{\delta}{1+\delta}} = \mathbb{E}\big[\zeta_1^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \sum_{n=1}^{\infty} n^{k+1} \mathbb{P}[N=n]^{\frac{\delta}{1+\delta}}.$$

Now applying Chebyshev inequality we get

$$\mathbb{E}[S^k] \leq \mathbb{E}\big[\zeta_1^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \sum_{n=1}^{\infty} n^{k+1} \frac{\mathbb{E}[N^l]^{\frac{\delta}{1+\delta}}}{n^{\frac{l\delta}{1+\delta}}} = \mathbb{E}\big[\zeta_1^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \mathbb{E}[N^l]^{\frac{\delta}{1+\delta}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{l\delta}{1+\delta}-(k+1)}} < \infty.$$

□

Proposition A.1. *We have that* $\overline{\mathcal{X}}$*, the closure in* $(\mathcal{H}, d_{\mathcal{H}})$ *of* $\mathcal{X}$*, is a compact set of* (Π, d)*.*

Proof. Using Lemma 6.2 we have that the number of paths in $\overline{\mathcal{X}}$ that cross $[a,b] \times \{t\}$ is finite. From this fact we get Proposition A.1 following the proof given in [NRS05]. □

Proposition A.2. *Let* $(\pi, t_0) \in \overline{\mathcal{X}}$ *then*

(i) *if* $t_0, \pi(t_0) \in \mathbb{R}$ *then* $(\pi, t_0) \in \mathcal{X}$*,*

(ii) *if* $t_0 = -\infty$ *then for each* $s \in \mathbb{R}$ *there exists* $v = (v_1, v_2)$ *such that* $v_2 \leq s$ *and* $\pi(t) = \pi^v(t)$ *for all* $t \geq s$*.*

Proof. Item (i) is an immediate consequence of the fact that if $(\pi, t_0) \in \mathcal{X}$ then $(t_0, \pi(t_0)) \in \mathbb{Z}^2$.

To proof of the item (ii) fix $s \in \mathbb{R}$ and let (π_n, t_n) be a sequence in $\mathcal{X}$ that $d((\pi_n, t_n), (\pi, t_0)) \to 0$ as $n \to \infty$. We can suppose that $t_n < s$ because we have that $t_n \to t_0 = -\infty$. Now for $S > s$ there exists $a < b \in \mathbb{R}$ such that $\pi_n(t) \in [a,b]$ for all $t \in [s,S]$ and all n large enough because $\Phi(\pi_n, t_n)$ converges uniformly to $\Phi(\pi, t_0)$. Take a partition take $\{r_i\}_{i=1}^k$ a partition of $[s,S]$ such that $0 < r_{i+1} - r_i < 1$. Again, there exists a finite number of paths in $\mathcal{X}$ that pass on $\bigcup_{i=1}^k [a,b] \times \{r_i\}$. Given that $\pi_n(r_i) \in \mathbb{Z}$ and converges to $\pi(r_i)$, we have $\pi_n(r_i) = \pi(r_i)$ for all n large enough. Using that and the linearity of the paths in $[r_i, r_{i+1}]$ we can get some M such that $\pi_n(t) = \pi_M(t)$ for all $n \geq M$ and $t \in [s,S]$; hence $\pi_M(t) = \pi(t)$ for all $t \in [s_1, s_2]$. Put

$$C := \big\{(\pi^v, v(2)) \in \mathcal{X}; v_2 \leq s \text{ and } \pi^v(t) = \pi(t) \text{ for all } t \in [s,S] \text{ for some } S > s\big\}.$$

For $(\pi^v, v(2)) \in C$ let us define $s_{\pi^v} := \sup\{S > s; \pi(t) = \pi^v(t) \text{ for all } t \in [s,S]\}$. Note that there exists some $(\pi^v, v(2)) \in C$ such that $s_{\pi^v} = \infty$ because there is only a finite number of paths in $\mathcal{X}$ that coincide with π in s and for all $S > s$ we get some $(\pi^u, u(2))$ in C such that $s_{\pi^u} \geq S$. □

Appendix B. Proof the technical estimates on coalescing times

This section is devoted to the proofs of Lemmas 4.2 and 4.3. The proofs rely on the use of a Skorohood's Representation of Y^m as done in [CFD09] and [CV14] and already introduced in this paper on Section 4. So recall from that section the definitions of $(B(s))_{s \geq 0}$ and $(S_i)_{i \geq 0}$.

Proof of Lemma 4.2. Let us start proving item (i). Define

$$C := \Big\{n \in [1, \nu^m_{(-\infty,0]}] \cap \mathbb{N}; (B(s))_{s \geq 0} \text{ visits } (-\infty, 0] \text{ in the interval } (S_{n-1}, S_n]\Big\}.$$

For $n \in C$ two things may occur:

1. $U_n(B(S_{n-1})) + B(S_{n-1}) = 0$. This implies that $n = \nu^m_{(-\infty,0]}$ because $(B(s))_{s \geq 0}$ visits $(-\infty, 0]$ in time interval $(S_{n-1}, S_n]$.

2. $U_n(B(S_{n-1}))+B(S_{n-1}) < 0$. In this case with probability bigger than a positive constant β, $(B(s))_{s\geq 0}$ will leave the interval $[U_n(B(S_{n-1}))+B(S_{n-1}), V_n(B(S_{n-1}))+B(S_{n-1})]$ by the left side, what implies that $n = \nu^m_{(-\infty,0]}$. Note that using the Strong Markov property and the fact that in $(S_{n-1}, S_n]$, $(B(s))_{s\geq 0}$ visits zero, β could be taken as the probability that a Standard Brownian motion leaves then interval $[-2Z, 1]$ by the left side; where Z is as defined in the Proposition 3.1 for two points.

Hence $\#C$ is stochastically bounded by a geometric random variable with parameter β. Since the Brownian motion is recurrent and $(-\infty, 0]$ is visited infinitely many times, we have that $\nu^m_{(-\infty,0]} < \infty$ almost surely.

Now we prove item (ii). For $m > M \in \mathbb{N}$ we get that $\nu^m_{(-\infty,M]} \leq \nu^m_{(-\infty,0]}$ hence by the item $i)$ we get

$$\mathbb{P}[\nu^m_{(-\infty,0]} < \infty] = \mathbb{P}[\nu^m_{(-\infty,M]} < \infty] = 1.$$

Note that

$$\begin{aligned}\mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0] \geq \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0, \nu^m_{(-\infty,0]} \neq \nu^m_{(-\infty,M]}] &= \sum_{k=1}^{M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0, Y^m_{\nu^m_{(-\infty,M]}} = k] \\ &= \sum_{k=1}^{M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0 | Y^m_{\nu^m_{(-\infty,M]}} = k] \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} = k].\end{aligned}$$

For all $1 \leq k \leq M$ by Strong Markov property and the translation invariance of the model we have that

$$\mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0 | Y^m_{\nu^m_{(-\infty,M]}} = k] = \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0].$$

Hence

$$\begin{aligned}\mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0] &\geq \sum_{k=1}^{M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} = k] \geq \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] \Big) \sum_{k=1}^{M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} = k] \\ &= \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] \Big) \mathbb{P}[\nu^m_{(-\infty,0]} \neq \nu^m_{(-\infty,M]}] \\ &\geq \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] \Big) \Big(\inf_{\tilde{m}>M} \mathbb{P}[\nu^{\tilde{m}}_{(-\infty,0]} \neq \nu^{\tilde{m}}_{(-\infty,M]}] \Big).\end{aligned}$$

From the description of the GRDF it is straighforward to verify that $\mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] > 0$ for all $k \geq 1$ and then $\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] > 0$. Now let us to prove that for an adequate M we have

$$\inf_{m>M} \mathbb{P}[\nu^m_{(-\infty,0]} \neq \nu^m_{(-\infty,M]}] > 0.$$

Note that

$$\begin{aligned}\mathbb{P}[\nu^m_{(-\infty,0]} = \nu^m_{(-\infty,M]}] = \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} \leq 0] &= \sum_{k=M+1}^{\infty} \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} \leq 0, Y^m_{\nu^m_{(-\infty,M]}-1} = k] \\ &= \sum_{k=M+1}^{\infty} \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} \leq 0 | Y^m_{\nu^m_{(-\infty,M]}-1} = k] \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}-1} = k].\end{aligned}$$

Again, using the Strong Markov property and the translation invariance we have

$$\mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} \leq 0 | Y^m_{\nu_{(-\infty,M]}-1} = k] = \mathbb{P}[Y^k_1 \leq 0].$$

For Z as defined in the Proposition 3.1 we get that

$$\begin{aligned}\mathbb{P}[Y_1^k \leq 0] &= \mathbb{P}[X_{\tau_1(u_k)}(u_k)(1) - X_{\tau_1(u_0)}(u_0)(1) \leq 0] \\ &\leq \mathbb{P}\Big[\big\{|X_{\tau_1(u_0)}(u_0)(1)| \geq \frac{k}{2}\big\} \cup \big\{|X_{\tau_1(u_k)}(u_k)(1) - k| \geq \frac{k}{2}\big\}\Big] \\ &\leq \mathbb{P}\Big[\big\{|X_{\tau_1(u_0)}(u_0)(1)| \geq \frac{k}{2}\big\}\Big] + \mathbb{P}\Big[\big\{|X_{\tau_1(u_k)}(u_k)(1) - k| \geq \frac{k}{2}\big\}\Big] \\ &\leq 2\mathbb{P}\big[Z \geq \frac{k}{2}\big] \leq 4\frac{\mathbb{E}[Z]}{k} \leq 4\frac{\mathbb{E}[Z]}{M}\,.\end{aligned}$$

Then

$$\mathbb{P}[\nu^m_{(-\infty,0]} = \nu^m_{(-\infty,M]}] \leq 4\frac{\mathbb{E}[Z]}{M} \sum_{k=M+1}^{\infty} \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}-1} = k] = 4\frac{\mathbb{E}[Z]}{M} := c_6.$$

Taking $M > 4\mathbb{E}[Z]$ we have that $\mathbb{P}[\nu^m_{(-\infty,0]} \neq \nu^m_{(-\infty,M]}] > 1 - c_6 > 0$ for all $m > M$.
Then we have that

$$\inf_{m>M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0] \geq \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0]\Big)(1 - c_6) > 0.$$

which completes the proof of (ii).
Now we prove (iii). Define

$$c_5 := \sup_{m\geq 1} \mathbb{P}[Y^m_{a_1} \neq 0].$$

By the item (ii) we have $c_5 < 1$. By definition $\mathbb{P}[Y^1_{a_1} \neq 0] \leq c_5$. The proof will follow by induction on k. Suppose that $\mathbb{P}[Y^1_{a_j} \neq 0,\ \text{for } j = 1, \ldots, k] \leq c_5^k$. Here we are going to assume that k is even, the case k odd is similar. Write

$$\begin{aligned}&\mathbb{P}[Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k+1] \\ &= \sum_{m\geq 1} \mathbb{P}[Y^1_{a_{k+1}} \neq 0, Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1] \\ &= \sum_{m\geq 1} \mathbb{P}[Y^1_{a_{k+1}} \neq 0 | Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1]\mathbb{P}[Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1].\end{aligned}$$

By Strong Markov property and the translation invariance of the model we have that

$$\mathbb{P}[Y^1_{a_{k+1}} \neq 0 | Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1] = \mathbb{P}[Y^m_{a_1} \neq 0] \leq c_5.$$

Hence

$$\begin{aligned}\mathbb{P}[Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k+1] &\leq c_5 \sum_{m\geq 1} \mathbb{P}[Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1] \\ &= c_5 \mathbb{P}[Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k] \leq c_5^{k+1}.\end{aligned}$$

□

Proof of Lemma 4.3. The proof is similar to the proof of Lemma 3.5 in [CV14]. By Skorohood Representation Theorem we have a Brownian motion $(B(s))_{s\geq 0}$ starting in 1 and stopping times $(S_n)_{n\geq 0}$, which could be both taken independent of $(Y^1_n)_{n\geq 1}$, such that

$$Y^1_n - Y^1_{n-1} \stackrel{d}{=} B(S_n) - B(S_{n-1})\,, \text{ for all } n \geq 1.$$

By Corollary 3.1 there exists a sequence of random variables $(Z_n)_{n\geq 1}$ such that $|Y_n^1 - Y_{n-1}^1| \leq 2Z_n$. Then

$$|B(S_n) - B(S_{n-1})| \stackrel{st}{\leq} 2Z_n\,, \text{ for all } n \geq 1. \tag{B.1}$$

Recall that $(S_i)_{i\geq 0}$ has the following representation.

$$S_0 := 0, S_i := \inf\Big\{s \geq S_{i-1}; B(s) - B(S_{i-1}) \notin \big(U_i(B(S_{i-1})), V_i(B(S_{i-1}))\big)\Big\} \tag{B.2}$$

where $\big\{(U_i(m), V_i(m)); m \in \mathbb{Z}, i \geq 1\big\}$ is a family of independent random vectors taking values in $\big((\mathbb{Z}_- - \{0\}) \times \mathbb{N}\big) \cup \big\{(0,0)\big\}$.
By (B.1) and (B.2) we have that

$$-2Z_n \stackrel{st}{\leq} \inf_{S_{n-1}\leq s\leq S_n}\{B(s) - B(S_{n-1})\} \leq \sup_{S_{n-1}\leq s\leq S_n}\{B(s) - B(S_{n-1})\} \stackrel{st}{\leq} 2Z_n \tag{B.3}$$

As in the proof of Lemma 4.2 consider the following random set

$$C := \Big\{n \in [1, \tau_{(-\infty,0]}] \cap \mathbb{N}; (B(s))_{s\geq 0} \text{ visits } (-\infty, 0] \text{ in the interval } (S_{n-1}, S_n]\Big\}.$$

Note that

$$-\sum_{i=1}^{|C|} 2Z_i \stackrel{st}{\leq} \inf_{0\leq s\leq \tau_{(-\infty,0]}} B(s)\,.$$

There exists a geometric random variable G (see the proof of item (i) of Lemma 4.2) such that G is stochastically above $|C|$. Then we have

$$-\sum_{i=1}^{G} 2Z_i \stackrel{st}{\leq} \inf_{0\leq s\leq \tau_{(-\infty,0]}} B(s).$$

Let us define the random variable R_1 as

$$R_1 := \begin{cases} -\sum_{i=1}^{G} 2Z_i;\ \text{if } |C| > 1 \\ 0;\ \text{otherwise,} \end{cases}$$

and $\widetilde{R}_0$ as an independent random variable such that $\widetilde{R}_0 \stackrel{d}{=} R_1|\{R_1 \neq 0\}$. Define

$$J_1 := \inf\{s \geq 0 : B(s) - B(0) = -(R_1 + \widetilde{R}_0)\}$$

which is clearly stochastically above S_{a_1}. Let $(\mathbb{B}(s))_{s\geq 0}$ be $(B(s))_{s\geq 0}$ translated to have $\mathbb{B}(0) = \widetilde{R}_0$, then $Y_{a_1} \neq 0$ is equivalent to $B(J_1) \neq 0$, indeed

$$Y_{a_1} \neq 0 \Leftrightarrow R_1 > 0 \Leftrightarrow \mathbb{B}(J_1) < 0\,.$$

From this point, it is straightforward to use an induction argument to build the sequence $\{R_j\}_{j\geq 1}$. At step j in the induction argument, we consider initially an excursion of $(\mathbb{B}(s))_{s\geq 0}$ in a time interval of size $(S_{a_j} - S_{a_{j-1}})$, and since $|Y^n_{a_{j-1}}| \leq R_{j-1}$ we can obtain R_j and define J_j using $(\mathbb{B}(s))_{s\geq 0}$ as before. By the strong Markov property of (Y_n^1), we obtain that the R_j's are independent and $Y_{a_j} \neq 0$ is equivalent to $B(J_j) \neq 0$. $\square$

Appendix C. Proof of the lemmas used to obtain condition T

Proof of Lemma 8.1. The proof follows from Proposition 3.1 and Lemma C.1 below, which is the same as Lemma 2.6 in [NRS05]. □

Lemma C.1. *Let $(S_n^x)_{n\geq 0}$ be a random walk with increments distributed as a random variable Z such that it starts from $x \in \mathbb{Z}_-$ at time 0. If $\mathbb{E}[|Z|^{k+2}] < \infty$ then $\{(S^x_{\nu^x_+})^k\}_{z\in\mathbb{Z}_-}$, where $\nu^x_+ = \inf\{n \geq 1; S^x_n > 0\}$, is uniformly integrable.*

Proof of Lemma 8.2. Recall from (3.2) the definition of the random variable

$$H(v) := \inf\left\{n \geq 1; \sum_{j=1}^{n} {}_{\{(v(1),v(2)+j) \text{ is open }\}} = K\right\},$$

where $v \in \mathbb{Z}^2$ and $K \in \mathbb{N}$ is such that $\mathbb{P}[W_v \leq K] = 1$. Put $H = H((0,0))$ and note that H has negative binomial distribution of parameters p and K, thus it has finite absolute moments of any order. If some path in $D(n\rho\sigma, n^2t\gamma)$ comes from $\mathbb{Z} \times \mathbb{Z}_-$ before crossing $R(0, 0; 2n\rho\sigma, 2n^2t\gamma)$ then $H(v) > n\rho\sigma$ for some $v \in \{-n\rho\sigma, \dots, n\rho\sigma\} \times \{0\}$, and

$$\begin{aligned}
&\mathbb{P}\big[H(v) > n\rho\sigma \text{ for some } v \in \{-n\rho\sigma, \dots, n\rho\sigma\} \times \{0\}\big] \\
&\leq 2n\rho\sigma\mathbb{P}[H > n\rho\sigma] \\
&\leq 2n\rho\frac{\mathbb{E}[H^2]}{(n\rho\sigma)^2} \to 0 \text{ as } n \text{ goes to infinity.}
\end{aligned}$$

The others paths in $D(n\rho\sigma, n^2t\gamma)$ come from points in with first component bigger than $2n\rho\sigma$ or smaller than $-2n\rho\sigma$ and second component between in $\{0, \dots, n^2t\gamma\}$. If from some $v = (v(1), v(2))$ with $v(1) > 2n\rho\sigma$ and $v(2) \in \{0, \dots, n^2t\gamma\}$ the path cross $R(0, 0; 2n\rho\sigma, 2n^2t\gamma)$ then $H(v) > v(1)$. In case that $v(1) < -2n\rho\sigma$ we have that $H(v) > -v(1)$. Then the probability that of one of these paths cross $R(0, 0; 2n\rho\sigma, 2n^2t\gamma)$ is bounded by

$$\begin{aligned}
2\sum_{j=1}^{2n^2t\gamma} \sum_{v\in\{(2n\rho\sigma,\infty)\cap\mathbb{Z}\}\times\{j\}} \mathbb{P}[H(v) > v(1)] &\leq 2(2n^2t\gamma)\sum_{i\geq 1}\mathbb{P}[H > i + 2n\rho\gamma] \\
&\leq 2(2n^2t\gamma)\sum_{i\geq 1}\frac{\mathbb{E}[H^6]}{(i + 2n\rho\sigma)^6} \\
&\leq 2(2n^2t\gamma)\sum_{i\geq 1}\frac{\mathbb{E}[H^6]}{i^3(2n\rho\sigma)^3} = \frac{C(t,\rho)}{n} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This completes the proof. □

Proof of the Lemma 8.3. The proof is analogous to the proof of Lemma 3.2 in [CV14]. For $l \in \mathbb{Z}$ consider $u_0 = (0, 0)$, $u_l = (l, 0)$ and the random times $(\tau_n(u_0))_{n\geq 1}, (\tau_n(u_l))_{n\geq 1}$ as introduced in Corollary 3.1 for the points u_0, u_l. Now define the following random walk

$$Y_0^l := l, Y_n^l := X_{\tau_n(u_0)}(u_0)(1) - X_{\tau_n(u_l)}(u_l)(1) \text{ for } n \geq 1.$$

Let $B^l(x,t)$ be the set of trajectories that remain in the interval $[l-x, l+x]$ during the time $[0,t]$. By the independence of the increments which implies the strong Markov property, we have that

$$\mathbb{P}(\nu_{x,y} > n^2\gamma t) \geq \mathbb{P}(\nu_{x,y,\rho^+} < n^2\gamma t \wedge \nu_{x,y}) \inf_{l\in\mathbb{Z}} \mathbb{P}(Y^l \in B^l(n\sigma\rho, n^2\gamma t)).$$

Note that

$$\inf_{l\in\mathbb{Z}} \mathbb{P}\big(Y^l \in B^l(n\sigma\rho, n^2\gamma t)\big) = 1 - \sup_{l\in\mathbb{Z}} \mathbb{P}\Big(\sup_{i\leq n^2\gamma t} |Y_i^l - l| \geq n\sigma\rho\Big).$$

Now

$$\limsup_{n\to\infty} \sup_{l\in\mathbb{Z}} \mathbb{P}\Big(\sup_{i\leq n^2\gamma t} |Y_i^l - l| > n\sigma\rho\Big),$$

is bounded above by

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{l\in\mathbb{Z}} \mathbb{P}\Big(\sup_{i\leq n^2\gamma t} &|X_{\tau_i(u_0)}(u_0)(1)| + |X_{\tau_i(u_l)}(u_l)(1) - l| > \frac{n\sigma\rho}{2}\Big) \\
&\leq 2 \limsup_{n\to\infty} \sup_{l\in\mathbb{Z}} \mathbb{P}\Big(\sup_{i\leq n^2\gamma t} |X_{\tau_i(u_0)}(u_0)(1)| > \frac{n\sigma\rho}{4}\Big) \\
&\leq 4\mathbb{P}\Big(N > \frac{\rho}{4\sqrt{t}}\Big) = 4e^{-\frac{\rho^2}{32\,t}},
\end{aligned}$$

where N is a standard normal random variable and the last inequality is a consequence of Donsker's Theorem, see also Lemma 2.3 in [NRS05]. Hence

$$\inf_{l\in\mathbb{Z}} \mathbb{P}\big(Y^l \in B^l(n\sigma\rho, n^2\gamma t)\big)$$

is bounded from below by a constant that depends only on t and ρ. So using Proposition 4.1 we obtain a constant $\tilde{C}(t,\rho)$ such that

$$\mathbb{P}(\nu_{x,y,\rho^+} < n^2\gamma t \wedge \nu_{x,y}) \leq \frac{\mathbb{P}(\nu_{x,y} > n^2\gamma t)}{\inf_{l\in\mathbb{Z}} \mathbb{P}\big(Y^l \in B^l(n\sigma\rho, n^2\gamma t)\big)} \leq \frac{\tilde{C}(t,\rho)\,|y-x|}{n}.$$

From the previous inequality we should follow the same steps as in the proof of Proposition 2.4 in [NRS05] to get an upper bound that do not depend on $|y-x|$. $\square$

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